Calculus of Variations 6: Hamilton's Principle

1. Start with a single particle of mass m, with position

$$q(t) = (q_1(t), q_2(t), q_3(t)), \tag{1}$$

velocity $v(t) = \dot{q}(t)$ and acceleration $a(t) = \ddot{q}(t)$ at time t. Suppose that the force $F = (F_1, F_2, F_3)$ acting the on the particle is conservative, with potential $U(q)$. Then

$$F = -\nabla U, \tag{2}$$

and Newton's second law is

$$\frac{\partial U}{\partial q_k} + m\ddot{q}_k = 0, \quad k = 1, 2, 3. \tag{3}$$

Think of the particle as a mechanical "system." The state, or *configuration* of the system at time t is $q(t)$. The set $\mathcal{C}$ of possible configurations is called the *configuration space.* In this case, $\mathcal{C} = \mathbf{R}^3$. The time-evolution of the configuration is governed by the system of equations (3).

2. Suppose that between times a and b, the particle traces a trajectory q in $\mathcal{C}$ with end-points

$$q(a) = \alpha \quad \text{and} \quad q(b) = \beta. \tag{4}$$

The potential energy of the particle is $U(q)$, and its kinetic energy

$$T(\dot{q}) = \frac{1}{2}m|\dot{q}|^2 = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2). \tag{5}$$

Consider the *action* functional

$$\begin{aligned} J(q) &= \int_a^b (T - U)\, dt \\ &= \int_a^b \left[\frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - U(q_1, q_2, q_3)\right] dt, \end{aligned} \tag{6}$$

defined on a domain $\mathcal{D}$ of smooth curves q satisfying the boundary conditions (4). The class $\mathcal{A}$ of admissible variations consists of smooth curves $h(t) = (h_1(t), h_2(t), h_3(t))$ that vanish at a and b. By the usual steps, the extremizing condition

$$\delta J(q, h) = 0, \quad \text{for all } h \in \mathcal{A}, \tag{7}$$

leads to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0, \quad k = 1, 2, 3, \tag{8}$$

where the Lagranigian is $L = T - U$. Note that $L_{q_k} = -U_{q_k}$ and $L_{\dot{q}_k} = m\dot{q}_k$. Plug these into (8) to get

$$\frac{\partial U}{\partial q_k} + m\ddot{q}_k = 0, \quad k = 1, 2, 3, \tag{9}$$

which are Newton's equations. Thus, you could say that

a. *The particle's trajectory $q(t)$ through $\mathbf{R}^3$ is determined by Newton's equations.*

Though you might prefer to say that

b. *The mechanical system's trajectory $q(t)$ through the configuration space $\mathcal{C}$ is an extremal of the action functional.*

This second characterization is a crude version of *Hamilton's principle,* a variational generalization of Newton's second law.

3. It is important to note that $q_1, \ldots, q_n$ are not necessarily the rectangular coordinates of a particle, but rather a set of quantities that describe the state of a mechanical system. Thus the q_k are called *generalized coordinates,* and the $\dot{q}_k$ the *generalized velocities.* As pointed out above, for the Lagrangian L in (6), $L_{\dot{q}_k} = m\dot{q}_k$. For this reason, the quantities

$$p_k = \frac{\partial L}{\partial \dot{q}_k}, \quad k = 1, \ldots, n, \tag{10}$$

are called the *generalized momenta.* And since $L_{q_k} = -U_{q_k}$, the L_{q_k} are called the *generalized forces.* The number n of generalized coordinated is called the number of *degrees of freedom* of the system.

4. **Example**: Consider a pendulum with string of length Λ and negligible mass and and a bob of mass m. As usual, θ is the angle made by the string and the pendulum. The kinetic energy is

$$T = \frac{1}{2}m(\Lambda\dot{\theta})^2,$$

and the potential,

$$U = mg\Lambda(1 - \cos\theta).$$

Hence the Lagrangian is

$$L(\theta, \dot{\theta}) = \frac{1}{2}m(\Lambda\dot{\theta})^2 - mg\Lambda(1 - \cos\theta). \tag{11}$$

By Hamilton's principle, the equation of motion is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0,$$

which reduces to

$$\ddot{\theta} + \frac{g}{\Lambda}\sin\theta = 0.$$

In this example the generalized coordinate is θ, the generalized velocity $\dot{\theta}$, the generalized momentum

$$p = \frac{\partial L}{\partial \dot{\theta}} = m\Lambda^2\dot{\theta},$$

and the generalized force

$$\frac{\partial L}{\partial \theta} = -mg\Lambda \sin\theta.$$

We take θ to be dimensionless. Note that the generalized velocity, momentum and force do not have the dimensions of their standard counterparts.

5. Since the Lagrangian in the previous example does not depend explicitly on time, the Hamiltonian

$$\begin{aligned} H &= -L + \dot{\theta}\, L_{\dot{\theta}} \\ &= \frac{1}{2}m(L\dot{\theta})^2 + mgL(1-\cos\theta) \\ &= T + U. \end{aligned} \tag{12}$$

is a first integral. Thus the total energy $T+U$ is constant, i.e. the energy is conserved.

6. Consider the planar motion of a body of mass m subject to the gravitational attraction F created by a body of mass M fixed at the origin. Let $x = (x_1, x_2)$ be the position of former body. Then

$$F = -\frac{GMm}{|x|^3}\, x.$$

The potential for the force field F is

$$U(x) = \frac{GMm}{|x|}, \tag{13}$$

and the kinetic energy

$$T(\dot{x}) = \frac{1}{2}m|\dot{x}|^2. \tag{14}$$

We can make things easier by introducing polar coordinates:

$$x_1 = r\cos\theta \quad \text{and} \quad x_2 = r\sin\theta.$$

In terms of r and θ, the potential energy is

$$U = U(r) = \frac{GMm}{r}, \tag{15}$$

and the kinetic,

$$T = T(r, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2). \tag{16}$$

The Lagrangian is

$$L(r,\dot r,\dot\theta) = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - \frac{GMm}{r}. \tag{17}$$

According to Hamilton's principle, the equations of motion are

$$\begin{cases} \dfrac{\partial L}{\partial r} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot r} = 0, \\ \dfrac{\partial L}{\partial \theta} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot\theta} = 0, \end{cases}$$

or

$$\begin{cases} mr\dot\theta^2 - \dfrac{GMm}{r^2} - \dfrac{d}{dt}(m\dot r) = 0, \\ \qquad\qquad\qquad -\dfrac{d}{dt}mr^2\dot\theta = 0. \end{cases}$$

In this example, the system has two degrees of freedom. The generalized coordinates are r and θ, the generalized velocities $\dot r$ and $\dot\theta$, and the generalized momenta $L_{\dot r} = mr\dot\theta^2$ and $L_{\dot\theta} = mr^2\dot\theta$.

7. Since the Lagrangian does not depend explicitly on time, the Hamiltonian

$$\begin{aligned} H &= -L + \dot r\, L_{\dot r} + \dot\theta\, L_{\dot\theta} \\ &= \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{GMm}{r} \\ &= T + U. \end{aligned} \tag{18}$$

is a first integral. Thus the total energy $T+U$ is constant, i.e. the energy is conserved.