Calculus of Variations 3: One Function of Several Variables

1. Let B be a blob in $\mathbf{R}^n$, and $x = (x_1, \ldots, x_n)$. For functions $u : \mathbf{R}^n \mapsto \mathbf{R}$, we define the functional

$$J(u) = \int_B L(x, u(x), \nabla u(x))\, dx.$$

The Lagangian L is a function of $2n+1$ arguments.

a. The domain $\mathcal{D}$ of J is a set of *smooth* functions u, i.e. each member of $\mathcal{D}$ has continuous partial derivatives to whatever order we require.

b. As in the single-variable case, we will usually impose the boundary condition on u,

$$u(x)\big|_{x \in \partial B} = f(x), \tag{1}$$

for some prescribed function $f : \mathbf{R}^n \mapsto \mathbf{R}$.

c. The class $\mathcal{A}$ of admissible variations consists of smooth functions $h : \mathbf{R}^n \mapsto \mathbf{R}$ that vanish on the boundary of B:

$$h(x)\big|_{x \in \partial B} = 0. \tag{2}$$

2. For $k = 1, \ldots, n$, set

$$u_{x_k} = \frac{\partial u}{\partial x_k}.$$

Define a vector field by

$$F = \left(\frac{\partial L}{\partial u_{x_1}}, \ldots, \frac{\partial L}{\partial u_{x_n}} \right).$$

Then the Gâteaux variation of J at u in the direction h is

$$\begin{aligned}
\delta J(u,h) &= \frac{d}{d\varepsilon} J(u + \varepsilon h)\big|_{\varepsilon = 0} \\
&= \int_B \left[\frac{\partial L}{\partial u} h + \sum_{k=1}^{n} \frac{\partial L}{\partial u_{x_k}} h_{x_k} \right] dx \\
&= \int_B \left[\frac{\partial L}{\partial u} h + F \cdot \nabla h \right] dx \\
&= \int_B \left[\frac{\partial L}{\partial u} - \operatorname{div} F \right] h\, dx + \int_{\partial B} hF \cdot \nu\, dS \\
&= \int_B \left[\frac{\partial L}{\partial u} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \frac{\partial L}{\partial u_{x_k}} \right] h\, dx. \qquad (3)
\end{aligned}$$

3. An extremal of J is a function $u \in D$ such that

$$\delta J(u,h) = 0, \tag{4}$$

for all admissible variations h. By (3), u is an extremal if it satisfies the Euler equation

$$\frac{\partial L}{\partial u} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \frac{\partial L}{\partial u_{x_k}} = 0. \tag{5}$$

As usual, we look for minimizers of J among its extremals.

4. **Example**: Let Γ be a smooth, closed curve in $\mathbf{R}^3$, with projection C in the x_1x_2-plane. (It's easiest to think of Γ as lying above the plane. This way, you get C by dropping Γ straight down onto the plane.) Let B be the region enclosed by C, so that $C = \partial B$. Select a function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ whose graph passes through Γ. Thus, for each point (x_1, x_2) on ∂B, $x_3 = f(x_1, x_2)$ is the elevation of the point above it on Γ.

   Suppose that you are to find the area of a smooth surface that has been "stretched" over Γ. If the surface is the graph of a function $x_3 = u(x_1, x_2)$, then

   a. u should be smooth, and

   b. u should satisfy the boundary condition

$$u\big|_{\partial B} = f. \tag{6}$$

   The surface area of the part of the graph of u lying over B is

$$\begin{aligned} J(u) &= \int_B \sqrt{1 + u_{x_1}^2 + u_{x_2}^2}\, dx_1 dx_2 \\ &= \int_B \sqrt{1 + |\nabla u|^2}\, dx_1 dx_2. \end{aligned}$$

The Lagrangian for J is

$$L = L(\nabla u) = \sqrt{1 + |\nabla u|^2}. \tag{7}$$

5. To find the surface bounded by Γ, of least area, is *Plateau's problem.* The solution comes in the form of a minimizer $u^* \in \mathcal{D}$ of J. Since u^* is to be found among the extremals, we solve the Euler equation (5) subject to the boundary condition (6). With the Lagrangian (7), Euler's equation is

$$\frac{\partial}{\partial x_1} \frac{u_{x_1}}{\sqrt{1 + |\nabla u|^2}} + \frac{\partial}{\partial x_2} \frac{u_{x_2}}{\sqrt{1 + |\nabla u|^2}} = 0, \tag{8}$$

   which simplifies to

$$(1 + u_{x_2}^2) u_{x_1 x_1} - 2 u_{x_1} u_{x_2} u_{x_1 x_2} + (1 + u_{x_1}^2) u_{x_2 x_2} = 0. \tag{9}$$

   This second-order, nonlinear, elliptic partial differential equation is called the *minimal surface equation.*. A solution u which satisfies the boundary condition (6) is an extremal of J.

6. **Example**: The *Laplacian* of $u : \mathbf{R}^n \mapsto \mathbf{R}$ is

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}.$$

For a region B in $\mathbf{R}^n$, let

$$J(u) = \frac{1}{2} \int_B |\nabla u|^2 \, dx.$$

The Euler equation for J is

$$\Delta u = 0. \tag{10}$$

This is *Laplace's equation*. For a given function $\varrho(x)$, let

$$J(u) = \int_B \left[\varrho(x) u + \frac{1}{2} |\nabla u|^2 \right] dx.$$

The Euler equation is

$$\Delta u = \varrho(x). \tag{11}$$

This is *Poisson's equation.* Both (10) and (11) are fundamental partial differential equations of applied mathematics.