Calculus of Variations 2: One Function of One Variable

1. We will look for minimizers of a functional of the form

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx, \tag{1}$$

defined on some domain $\mathcal{D}$ of smooth functions y which satisfy the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta. \tag{2}$$

For this problem, the class $\mathcal{A}$ of admissible variations consists of smooth functions h which vanish at the endpoints of the interval, i.e.

$$h(a) = h(b) = 0. \tag{3}$$

In the applied calculus of variations, it is not always profitable to split hairs over regularity. We will instead proceed formally, assuming that L, y and h are as smooth as we need them to be. The plan is

a. Compute the Gâteaux variation $\delta J(y, h)$ for $y \in \mathcal{D}$ and $h \in \mathcal{A}$.

b. Solve the equation $\delta J(y, h) = 0$ for extremals y.

c. Identify the miminizers among the extremals.

In calculus, (c) is usually done with a derivative test. There is a second derivative test in the calculus of variations, but for now we'll ignore it, appealing instead to geometric or physical intuition to establish the *bona fides* of a minimizer.

2. We're now ready for steps (a) and (b). For y in $\mathcal{D}$ and all h in $\mathcal{A}$,

$$\begin{aligned}\delta J(y, h) &= \frac{d}{d\varepsilon}\, J(y + \varepsilon h)\,\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \int_a^b L(x, y + \varepsilon h, y' + \varepsilon h')\, dx\,\Big|_{\varepsilon=0} \\ &= \int_a^b [L_y(x, y, y')h + L_{y'}(x, y, y')h']\, dx. \end{aligned} \tag{4}$$

Integrate by parts to get rid of the derivative on h'.

$$\delta J(y, h) = \int_a^b \left[L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y')\right] h\, dx + L_{y'}(x, y, y')h\Big|_{x=a}^{x=b}. \tag{5}$$

By (2), the boundary terms vanish, leaving

$$\delta J(y, h) = \int_a^b \left[L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y')\right] h\, dx. \tag{6}$$

Thus,

$$\delta J(y,h) = 0,$$

for all admissible variations h if y satisfies the *Euler equation,*

$$L_y(x, y(x), y'(x)) - \frac{d}{dx} L_{y'}(x, y(x), y'(x)) = 0. \tag{7}$$

We have reduced the problem of finding extremals to solving the two-point boundary value problem,

$$\begin{cases} L_y(x,y,y') - \frac{d}{dx} L_{y'}(x,y,y') = 0, & \text{for } a < x < b, \\ y(a) = \alpha, \\ y(b) = \beta. \end{cases} \tag{8}$$

3. **Example**: Consider the arclength functional J defined in the last set of notes. For y in the domain

$$\mathcal{D} = \{y \in C^1[0,1] \mid y(0) = 0,\ y(1) = 1\},$$

we set

$$J(y) = \int_0^1 \sqrt{1 + y'(x)^2}\, dx. \tag{9}$$

$J(y)$ is the length of a smooth curve $y = y(x)$ drawn from $(0,0)$ to $(1,1)$. The shortest path between these two points is the graph of a minimizer y^* of J. That minimizer is to be sought among the extremals of J. As we just saw, these extemals are solutions to the boundary value problem:

$$\begin{cases} L_y - \frac{d}{dx} L_{y'} = 0, \\ y(0) = 0, \\ y(1) = 1. \end{cases}$$

The Lagrangian is

$$L(y') = \sqrt{1 + y'^2},$$

and hence the Euler equation,

$$\frac{d}{dx} L_{y'} = \frac{d}{dx} \frac{y'(x)}{\sqrt{1 + y'(x)^2}} = 0. \tag{10}$$

It follows that

$$\frac{y'(x)}{\sqrt{1 + y'(x)^2}} = C, \tag{11}$$

for some constant C. From (11), $y'(x) = A$, where $A^2 = C^2/(1 - C^2)$. So,

$$y(x) = Ax + B.$$

for constants A and B. By the boundary conditions, $A = 1$ and $B = 0$. Thus the sole extremal of J is $y(x) = x$. Clearly, this is also the sought-after minimizer, $y^*(x)$.

4. The *Hamiltonian* corresponding to a Lagrangian L is

$$H = -L(x, y, y') + y' \, L(x, y, y').$$

5. A *first integral* $g(x, y, y')$ of a second order differential equation $F(x, y, y', y'') = 0$, is a conserved quantity, i.e.

$$g(x, y(x), y'(x)) = \text{constant},$$

if y satisfies the differential equation. We can often write down a first integral for the Euler equation.

a. If $L = L(y, y')$, then the Hamiltonian

$$H = -L(y, y') + y' \, L_{y'}(y, y'),$$

is a first integral of the Euler equation.

b. If $L = L(x, y')$, then

$$L_{y'}(x, y'),$$

is a first integral.

c. If $L = L(x, y)$, then the Euler equation reduces to the algebraic equation

$$L_y(x, y) = 0.$$

6. **Example**: (The Brachistichrone Problem) A bead of mass m, initially at rest, slides without friction along the curve $y = y(x)$ from (a, α) to (b, β), where $a < b$ and $\alpha > \beta$. The only force acting on the bead is gravity. We'll use the calculus of variations to find the curve y^* that minimizes the bead's transit time T. Let s be arclength along the the curve, and S the curve's total length. Then

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{1 + y'(x)^2} \, dx, \end{aligned}$$

so that

$$\frac{ds}{dx} = \sqrt{1 + y'(x)^2}. \tag{12}$$

Let v be the speed of the bead. The bead is initially at rest. Hence there is no kinetic energy at time $t = 0$, and the total energy is equal to the potential energy, $mg\alpha$. Since energy is conserved,

$$\frac{1}{2}mv^2 + mgy = mg\alpha.$$

Consequently,

$$v = \frac{ds}{dt} = \sqrt{2g(\alpha - y)}. \tag{13}$$

By (12) and (13),

$$\begin{aligned} T &= \int_0^T dt \\ &= \int_0^S \frac{dt}{ds}\, ds \\ &= \int_a^b \frac{dt}{ds}\frac{ds}{dx}\, dx \\ &= \int_a^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2g(\alpha - y(x))}}\, dx \end{aligned} \tag{14}$$

The integral (14) gives T as a functional of y. The extremals of T are those solutions y to the Euler equation that satisfy the boundary conditions (2). The Lagrangian is

$$L(y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2g(\alpha - y)}}.$$

The Hamiltonian is a first integral. So,

$$H(y, y') = -\frac{\sqrt{1 + y'^2}}{\sqrt{2g(\alpha - y)}} + \frac{y'^2}{\sqrt{1 + y'^2}\,\sqrt{2g(\alpha - y)}} = C, \tag{15}$$

for a constant C. Multiply (15) by

$$\sqrt{1 + y'^2}\,\sqrt{2g(\alpha - y)},$$

absorb the $2g$ into a new constant A and square both sides of the resulting equation to get

$$1 = A^2(1 + y'^2)\,(\alpha - y). \tag{16}$$

Thus y satisfies the first-order differential equation

$$y' = \pm\sqrt{\frac{1 - A^2(\alpha - y)}{A^2(\alpha - y)}}. \tag{17}$$

Since bead should slide down the wire, we take the negative square root in (17). Next, set

$$A^2(\alpha - y) = \sin^2\theta, \tag{18}$$

where $0 \leq \theta \leq \pi/2$. From here, a little more calculus gets you to

$$x = \frac{1}{A^2}\left[\theta - \frac{\sin 2\theta}{2}\right] + B, \tag{19}$$

where B is a constant. (18) and (19) are parametric equations for a *cycloid.* The constants A and B are determined by the boundary conditions (2).

7. It should be clear that the function y defined in (19) is an extremal of T. We have yet to prove that y is a minimizer.