Linearity, Superposition and Classification

1. A partial differential operator (PDO) L is linear if for any functions u and v and and scalars c,

$$L[u + cv] = Lu + cLv.$$

If L is a linear PDO, the equation

$$Lu = f,$$

is homogeneous if $f \equiv 0$, and inhomogeneous otherwise. The *order* of a partial differential equation is the order of the highest derivative appearing in it. The wave, heat and Laplace equations are second-order, linear, homogeneous partial differential equations. The inviscid Burgers' equation

$$u_t + uu_x = 0, \tag{1}$$

is a first-order, nonlinear PDE. The Korteweg-deVries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{2}$$

is third-order, nonlinear.

2. **Superposition:** If $u_1, \dots, u_n$ satisfy the linear, homogeneous equation

$$Lv = 0, \tag{3}$$

then any linear combination $u = c_1 u_1 + \cdots + c_n u_n$ also satisfies it.

3. Extensions: If $\{u_k\}_{k=1}^{\infty}$ satisfy (3) and

$$u = \sum_{k=1}^{\infty} c_k u_k,$$

converges "well enough," then u also satisfies (3). If $u(x, \alpha)$ satisfy (3) for all α in some interval I and

$$u(x) = \int_I c(\alpha) u(x, \alpha)\, d\alpha,$$

converges well enough, then u also satisfies (3).

4. Consider second-order, linear PDO in two independent variables. Associate with a t-derivative the symbol τ, with an x-derivative ξ and with a y-derivative η. The heat operator is

$$H = \frac{\partial}{\partial t} - k\Delta,$$

where $k > 0$. We thus associate with it the symbol $\tau - k\xi^2$, suggesting a parabola in the $\xi\tau$-plane. For this reason, the heat operator is called parabolic. The heat equation is a parabolic PDE. The Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

with symbol $\xi^2 + \eta^2$. This suggests an ellipse (a circle in this case) in the $\xi\eta$-plane. Thus the Laplacian is an elliptic PDO. Laplace's equation is an elliptic PDE. Finally, the wave operator

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2},$$

has symbol $\tau^2 - c^2\xi^2$ and is thus a hyperbolic operator. The wave equation is a hyperbolic PDE.

5. We won't go into the details of classification, but there are a few points to bear in mind:

a. Parabolic equations govern phenomena (e.g. diffusion) characterized by smoothing, spreading flow. The heat operator is the archetypal parabolic operator.

b. Elliptic equations govern equilibrium, energy-minimizing states. The Laplacian Δ is the archetypal elliptic operator.

c. Hyperbolic equations govern "distrubance preserving" phenomena such as travelling waves and shocks. The D'Alembertian is the archetypal hyperbolic operator.

d. Many PDOs (or PDEs) do not fall into one of the three categories. For example, the Tricomi operator (from gas dynamics)

$$L = y\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

is hyperbolic for $y < 0$ and elliptic for $y > 0$.

e. Classification is harder with more independent variables, with higher-order PDE and with systems of equations.

f. There are various refinements and extensions of the linear-nonlinear, homogeneous-inhomogeneous, parabolic-elliptic-hyperbolic classifications. You might, for example, have a "quasilinear elliptic" equation, or a system of equations that is "symmetric hyperbolic." The inviscid Burgers equation (1) is an example of a *first-order, quasilinear, scalar conservation law.* Since the KdV equation (2) is nonlinear and admits wave solutions, it is sometimes referred to as a *nonlinear wave equation.*