The Gâteaux Variation of a Functional

1. Let X be a linear space over $\mathbf{R}$ equipped with norm $\| \, \|$. A real-valued *functional* J on X is just a function mapping X to $\mathbf{R}$. We write

$$J : X \mapsto \mathbf{R}. \tag{1}$$

The set $\mathcal{D}$ of points $x \in X$ for which $J(x)$ is defined is called the *domain* of J. Instead of (1), you can write

$$J : \mathcal{D} \mapsto \mathbf{R},$$

or

$$J : \mathcal{D} \subset X \mapsto \mathbf{R},$$

if you wish to advertise the role of the domain in the defintion of J.

2. **Example**: Let $J : C[0,1] \mapsto \mathbf{R}$ by

$$J(x) = \int_0^1 x(t)e^{-t}\,dt.$$

3. **Example**: Let $J : C^1[a,b] \mapsto \mathbf{R}$ by

$$J(x) = \int_a^b \dot{x}(t)^2\,dt.$$

4. **Example**: Define the functional J on the domain

$$\mathcal{D} = \{y \in C^1[0,1] \mid y(0) = 0,\; y(1) = 1\},$$

by

$$J(y) = \int_0^1 \sqrt{1 + y'(x)^2}\,dx. \tag{2}$$

$J(y)$ is the length of a smooth curve $y = y(x)$ drawn from $(0,0)$ to $(1,1)$.

5. Many of the funtionals in the calculus of variations are of the form

$$J(y) = \int_a^b L(x, y(x), y'(x))\,dx, \tag{3}$$

for y defined on some domain $\mathcal{D} \subseteq C^1[a,b]$. The function L is called the *Lagrangian.*

6. Let $J : \mathcal{D} \subset X \mapsto \mathbf{R}$ be a functional. A point $x^* \in \mathcal{D}$ is a *local minimizer* of J if there is an $r > 0$ such that $J(x^*) \le J(x)$ for all $x \in \mathcal{D} \cap B(x^*, r)$. The number $J(x^*)$ is a *local minimum* of J.

7. Let f be a smooth function taking $\mathbf{R}^n$ to $\mathbf{R}$. The derivative of f at $x \in \mathbf{R}^n$ in the direction $h \in \mathbf{R}^n$ is

$$\begin{aligned} D_h f(x) &= \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon h) - f(x)}{\varepsilon} \\ &= \frac{d}{d\varepsilon} f(x+\varepsilon h)\Big|_{\varepsilon=0}. \end{aligned} \tag{4}$$

Looking to (4), we define the *Gâteaux variation* (or first variation, or Gâteaux derivative) of the functional J at $x \in \mathcal{D}$ in the direction h to be

$$\delta J(x,h) = \frac{d}{d\varepsilon} J(x+\varepsilon h)\Big|_{\varepsilon=0}. \tag{5}$$

Don't worry about the name and notation. The Gâteaux variation is just a directional derivative. It would be perfectly reasonable to write $D_h J(x)$ instead of $\delta J(x,h)$.

8. You have to take some care in the choice of directions h used in the calculation of the Gâteuax variation. By the definition (4), you may only take those vectors $h \in X$ for which $x + \varepsilon h$ lies in $\mathcal{D}$ for ε small. Such vectors are called *admissible variations*. For a functional J, we denote by $\mathcal{A}$ its class of admissible variations.

9. At a *critical point* x, of a smooth function f, the derivative in every direction is zero: Thus,

$$D_h f(x) = 0, \quad \text{for all } h \text{ in } \mathbf{R}^n. \tag{6}$$

The same notion appears in the calculus of variations, though it goes by a different name. At an *extremal* $x \in \mathcal{D}$ of a functional J,

$$\delta J(x,h) = 0, \quad \text{for all } h \text{ in } \mathcal{A}. \tag{7}$$

The functional J is said to be *stationary* at the extremal x.

10. In calculus, you learned to seek the local minimizers of f among its critical points. In the calculus of variations, one seeks the local minimizers of a functional J among its extremals. You find those extremals by solving (6) for x.

11. Here is a summary of the correspondences between the ideas, terminology and notation of the calculus of variations (on the left) and those of calculus (on the right).

A functional $J : \mathcal{D} \subset X \mapsto \mathbf{R} \iff$ A function $f : \mathbf{R}^n \mapsto \mathbf{R}$

An admissible variation $h \in \mathcal{A} \iff$ A direction vector $h \in \mathbf{R}^n$

The Gâteaux variation $\delta J(x,h) \iff$ The directional derivative $D_h f(x)$

An extremal $x \in \mathcal{D}$ of $J \iff$ A critical point $x \in \mathbf{R}^n$ of f