Balance Laws 1

1. A fluid is flowing in a region in $\mathbf{R}^3$. Let $v(x,t)$ and $\varrho(x,t)$ be the velocity vector and mass density function at $x \in \mathbf{R}^3$ and time t. Note that v is a time dependent vector field on $\mathbf{R}^3$ and that ϱ is a scalar-valued function. Let ∂B be a smooth, closed surface bounding a volume B.

2. Let $M = M(t)$ be the mass of fluid in B at time t. Clearly,

$$\begin{aligned}\frac{dM}{dt} = &- \{\text{Net flux of fluid out of } B \text{ though } \partial B\} \\ &+ \{\text{Net rate at which fluid is created in } B\}. \end{aligned} \tag{1}$$

When computing the net flux out of B, we count as negative flow into B and positive flow out. Similarly, the net rate at which fluid is produced is the rate of fluid creation minus the rate of fluid destruction.

3. The mass in B is

$$M = \int_B \varrho \, dx. \tag{2}$$

Hence

$$\frac{dM}{dt} = \frac{d}{dt} \int_B \varrho \, dx. \tag{3}$$

4. Let P be an infinitesimally small patch of ∂B with area dS. We take P so small as to be virtually flat. Thus ν is constant over P. By the same token, we can take v and ϱ to be constant over P between times t and $t + dt$. Let θ be the angle between ν and v. Over the time period $[t, t+dt]$, the net amount of fluid flowing through P is

$$dm = \varrho |v| \, dt \cos\theta \, dS = \varrho v \cdot \nu \, dS dt.$$

Thus the net rate of fluid flow (the flux) through P is

$$\frac{dm}{dt} = \varrho v \cdot \nu \, dS.$$

Integrate the above expression over B to obtain

$$\{\text{Net flux of fluid out of } B \text{ though } \partial B\} = \int_{\partial B} \varrho v \cdot \nu \, dS. \tag{4}$$

5. Let $f(x,t)$ be the net rate per unit volume at which fluid is produced at the point x and time t. Thus

$$\{\text{Net rate at which fluid is created in } B\} = \int_B f \, dx. \tag{5}$$

6. Plug (3), (4) and (5) into the balance law (1):

$$\frac{d}{dt}\int_B \varrho\,dx + \int_{\partial B} \varrho v\cdot\nu\,dS = \int_B f\,dx. \tag{6}$$

This is the balance law in its most general form. If ϱ and v are smooth, we can move the time derivative into the first integral and apply the divergence theorem to the second, yielding

$$\frac{d}{dt}\int_B \varrho\,dx = \int_B \varrho_t\,dx,$$

and

$$\int_{\partial B} \varrho v\cdot\nu\,dS = \int_B \operatorname{div}(\varrho v)\,dx.$$

We can thus rewrite the balance law as

$$\int_B [\varrho_t + \operatorname{div}(\varrho v)]\,dx = \int_B f\,dx. \tag{7}$$

Equations (6) and (7) are the integral forms of the balance law. If (7) holds for every bounded volume B, then it must hold pointwise. Thus,

$$\varrho_t + \operatorname{div}(\varrho v) = f, \tag{8}$$

which is the balance law in differential form. Finally, if fluid is neither created nor destroyed, then $f \equiv 0$, and (8) becomes the *conservation law*

$$\varrho_t + \operatorname{div}(\varrho v) = 0. \tag{9}$$

(9) is usually called the *continuity equation.*

7. The mass flux of the fluid through ∂B is

$$\int_{\partial B} \varrho v\cdot\nu\,dS.$$

The vector ϱv is called the *flux density vector,* or simply the *flux vector.* In general, if the flux of some quantity (call it "stuff") across a patch of area dS is $J\cdot\nu\,dS$, then J is the flux density vector for the flow of stuff. Note that J has dimensions

$$[J] = [\text{stuff}] \times \text{area}^{-1} \times \text{time}^{-1}.$$

8. The general case: If "stuff" flowing in $\mathbf{R}^3$ with density ϱ and flux density vector J is manufactured at net rate per unit volume f, then the balance law in integral form is

$$\frac{d}{dt}\int_B \varrho\,dx + \int_{\partial B} J\cdot\nu\,dS = \int_B f\,dx. \tag{10}$$

Moving the time derivative inside and applying the divergence theorem gives us

$$\int_B \varrho_t \, dx + \int_B \operatorname{div} J \, dx = \int_B f \, dx. \tag{11}$$

If this holds for all closed volumes B, we obtain the differential form of the balance law

$$\varrho_t + \operatorname{div} J = f, \tag{12}$$

and the conservation law

$$\varrho_t + \operatorname{div} J = 0, \tag{13}$$

when $f \equiv 0$.

9. **Example**: Let ϱ be the mass density of a gas that moves by *diffusion* alone. This means that the gas simply spreads from regions of higher to regions of lower concentration. Since $-\nabla\varrho$ is the direction in which the density decreases most rapidly, you could reasonably assume that

$$J = -D\nabla\varrho, \tag{14}$$

for some positive constant D with $[D] = \text{area} \times \text{time}^{-1}$. Plug this into (14) to get

$$\varrho_t - D\Delta\varrho = 0. \tag{15}$$

This is the simplest partial differential equation governing diffusion. Since it was derived by Fourier in his study of heat flow, it is usually called the *heat equation.* Chemists refer to (14) and (15) as Fick's laws.

10. The constant D is called the *diffusion coefficient.* In more sophisticated models, D is allowed to depend on ϱ, x or t. For example, if $D = D(\varrho)$, then (14) becomes

$$J = -D(\varrho)\nabla\varrho, \tag{16}$$

and (15),

$$\varrho_t - \operatorname{div}(D(\varrho)\nabla\varrho) = 0. \tag{17}$$

This is a *nonlinear heat equation.*

11. Equations (14) and (16) are not laws of nature, but sensible assumptions relating quantities that appear in the mathematical model. Such equations are called *constitutive.*