Advanced Calculus 2

1. The derivative: Let $\xi \in \mathbf{R}^m$ have components ξ_j and $x\colon \mathbf{R}^m \mapsto \mathbf{R}^n$ by

$$x(\xi) = \begin{bmatrix} x_1(\xi) \\ \vdots \\ x_n(\xi) \end{bmatrix}.$$

We take ξ to be continuously differentiable, i.e. each of the components x_j has continuous first partial derivatives. The *derivative* of x is the $n \times m$ matrix of partial derivatives

$$x'(\xi) = \Big(x_{i_{\xi_j}}(\xi) \Big) = \begin{bmatrix} x_{1_{\xi_1}}(\xi) & \cdots & x_{1_{\xi_m}}(\xi) \\ \vdots & \ddots & \vdots \\ x_{n_{\xi_1}}(\xi) & \cdots & x_{n_{\xi_m}}(\xi) \end{bmatrix}. \tag{1}$$

2. If $g : \mathbf{R}^n \mapsto \mathbf{R}^k$. The *chain rule* asserts that the composition $f = g \circ x \colon \mathbf{R}^m \mapsto \mathbf{R}^k$ has derivative

$$f'(\xi) = g'(x(\xi))x'(\xi).$$

3. Let c be a point in $\mathbf{R}^m$. Taylor's theorem tells us that

$$x(\xi) = x(\xi) + x'(c)(\xi - c) + o(|\xi - c\,|). \tag{2}$$

4. Let $B \subseteq \mathbf{R}^m$, and $x : \mathbf{R}^m \mapsto \mathbf{R}^m$. The *image of* B *under* x is the set

$$x(B) = \{y \in \mathbf{R}^n \mid y = x(\xi) \text{ for some } \xi \in B\}.$$

Let A be an $m \times m$ matrix. You can think of A as a linear function taking $\mathbf{R}^m$ to $\mathbf{R}^m$. We denote by AB the image of B under A. In a word,

$$AB = \{y \in \mathbf{R}^m \mid y = A\xi \text{ for some } \xi \in B\}.$$

5. **Proposition**: Let $\mu(\cdot)$ be m-dimensional measure, B a subset of $\mathbf{R}^m$ and A an $m \times m$ matrix. Then

$$\mu(AB) = |\det A\,|\,\mu(B). \tag{3}$$

So when you map $\mathbf{R}^m$ to $\mathbf{R}^m$ *linearly,* the m-dimensional volume of a set is magnified by a factor $|\det A\,|$.

6. Let $\xi : \mathbf{R}^m \mapsto \mathbf{R}^m$ and B be a "small" box in $\mathbf{R}^m$, of volume centered at the point x. Since B is small,

$$x(\xi + h) \approx x(c) + x'(c)(\xi - c), \tag{4}$$

for every $\xi \in B$. This tells us that we can approximate the action of x on B in three steps. These are

a. Subtract c from each point ξ in B. This is a translation.

b. Multiply by the matrix $x'(c)$.

c. Add $x(c)$. This is also a translation.

Translation does not change the measure of a set in $\mathbf{R}^m$. Multiplcation by $x'(c)$ magnifies the measure by the factor $|\det\, x'(c)|$. Thus, by (4) and (a)-(c),

$$\mu(x(B)) \approx |\det\, x'(c)|\, \mu(B), \tag{5}$$

for a *small* box B centered at c. Let ΔV_ξ be the volume of B in $\mathbf{R}^m_x$ and ΔV_x the volume of the image $x(B)$ in $\mathbf{R}^m_x$. With this notation, (5) is

$$\Delta V_x \approx |\det\, x'(c)|\, \Delta V_\xi, \tag{6}$$

7. If we replace the small box centered at c with an "infinitesimally" small box about the point ξ, (6) becomes

$$dV_\xi = |\det\, x'(x)|\, dV_\xi,$$

or, in our usual notation,

$$dx = |\det\, x'(x)|\, d\xi. \tag{7}$$

8. The determinant of $x'(\xi)$ is called the *Jacobian,* and is denoted

$$\det\, x'(\xi) = J(\xi) = \frac{\partial(x_1, \ldots, x_m)}{\partial(\xi_1, \ldots, \xi_m)}.$$

Thus

$$dx = |J(\xi)|\, d\xi = \left| \frac{\partial(x_1, \ldots, x_m)}{\partial(\xi_1, \ldots, \xi_m)} \right| d\xi. \tag{8}$$

9. Change of variable in several dimensions: Let $B \subset \mathbf{R}^m$, and $f : \mathbf{R}^m \mapsto \mathbf{R}$. Suppose that we are to evaluate the integral

$$I = \int_B f(x)\, dx,$$

by making a change of variable

$$\begin{cases} x_1 = x_1(\xi_1, \ldots, \xi_m), \\ \qquad\vdots \\ x_m = x_m(\xi_1, \ldots, \xi_m). \end{cases} \tag{9}$$

We assume the transformation (9) to be invertible, that is, we can for the x_j as functions of ξ:

$$\begin{cases} \xi_1 = \xi_1(x_1, \ldots, x_m), \\ \qquad\vdots \\ \xi_m = \xi_m(x_1, \ldots, x_m). \end{cases} \tag{10}$$

We can set $x = (x_1, \ldots, x_m)$ and write (9) and (10) succinctly as $x = x(\xi)$ and $\xi = \xi(x)$ respectively. The recasting of the integral proceeds by three steps:

a. Write the integrand as a function of ξ: $f(x) = f(x(\xi))$.

b. Replace the domain of integration $B \subset \mathbf{R}_x^m$ with $\xi(B) \subset \mathbf{R}_\xi^m$.

c. Replace dx with $|J(\xi)|\, d\xi$.

Thus,

$$I = \int_B f(x)\, dx = \int_{\xi(B)} f(x(\xi)) \left| \frac{\partial(x_1, \ldots, x_m)}{\partial(\xi_1, \ldots, \xi_m)} \right| d\xi. \tag{11}$$