Advanced Calculus

1. **Definition:** The Cartesian product $A \times B$ of sets A and B is

$$A \times B = \{(a,b) |\ a \in A,\ b \in B\}.$$

This generalizes in the obvious way to higher Cartesian products:

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) |\ a_1 \in A_1, \ldots, a_n \in A_n\}.$$

2. **Example:** The real line $\mathbf{R}$, the plane $\mathbf{R^2} = \mathbf{R} \times \mathbf{R}$, the n-dimensional real vector space $\mathbf{R^n}$, $\mathbf{R} \times [\mathbf{0}, \mathbf{T}]$, $\mathbf{R^n} \times [\mathbf{0}, \mathbf{T}]$ etc.

3. **Definition:** The dot product (or inner or scalar product) of vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R^n}$ is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

4. **Definition:** The magnitude (or 2-norm) of a vector $x = (x_1, \ldots, x_n)$ in $\mathbf{R^n}$ is

$$|x| = \left[\sum_{i=1}^{n} x_i^2 \right]^{\frac{1}{2}} = (x \cdot x)^{\frac{1}{2}}.$$

5. **Definition:** We write $f : A \mapsto B$ if f is a function whose domain is a subset of A and whose range is a subset of B. Let

$$x = (x_1, \ldots, x_n)$$

be a point in $\mathbf{R^n}$. So, if $f(x) = f(x_1, \ldots, x_n)$ is a function that has its domain in $\mathbf{R^n}$, we write

$$f : \mathbf{R^n} \mapsto \mathbf{R}.$$

So, for example, if

$$f(x_1, x_2, x_3, x_4) = x_2 x_4 \sin{(x_1^2 + x_3^2)},$$

then

$$f : \mathbf{R^4} \mapsto \mathbf{R}.$$

We say that f takes $\mathbf{R^4}$ to $\mathbf{R}$.

6. This notion is easily generalized to functions from $\mathbf{R^n}$ to $\mathbf{R^m}$. Let $x \in \mathbf{R^n}$ and $f_1(x), \ldots, f_m(x)$ be functions taking $\mathbf{R^n}$ to $\mathbf{R}$. Then we can define a function

$$f : \mathbf{R^n} \mapsto \mathbf{R^m},$$

by

$$f(x) = (f_1(x), \ldots, f_m(x)).$$

7. **Example:** For $t \in \mathbf{R}$, let $f_1(t) = \cos t$, $f_2(t) = \sin t$ and $f : \mathbf{R} \mapsto \mathbf{R^2}$ by

$$f(t) = (f_1(t), f_2(t)) = (\cos t, \sin t).$$

As t ranges through the real numbers, $f(t)$ traces the unit circle counterclockwise in $\mathbf{R^2}$.

8. Let $F_i : \mathbf{R^n} \mapsto \mathbf{R}$ for $i = 1, \ldots, n$. Then

$$F(x) = (F_1(x), , \ldots, F_n(x)),$$

defines a vector field on $\mathbf{R^n}$. In other words, $F(x) : \mathbf{R^n} \mapsto \mathbf{R^n}$. So, for example, a mass M at the origin $(0, 0, 0)$ in $\mathbf{R^3}$ creates a gravitational field

$$F(x) = -\frac{GM}{|x|^3}\, x,$$

at $x = (x_1, x_2, x_3)$. So F is a vector field on $\mathbf{R^3}$.

9. **Definition:** Let $f : \mathbf{R^n} \mapsto \mathbf{R}$. The gradient of f at x is the vector

$$\nabla f(x) = \left(f_{x_1}(x), \ldots, f_{x_n}(x)\right),$$

where

$$f_{x_i} = \frac{\partial f}{\partial x_i}.$$

10. **Definition:** Let $F : \mathbf{R^n} \mapsto \mathbf{R^n}$ be a vector field on $\mathbf{R^n}$ and $f : \mathbf{R^n} \mapsto \mathbf{R}$. If

$$-\nabla f(x) = F(x),$$

then f is a potential function (or simply, a potential) for F.

11. Let x and h be vectors in $\mathbf{R^n}$. Let $f : \mathbf{R^n} \mapsto \mathbf{R}$. Recall that the derivative of f at x in the direction h is

$$D_h f(x) = \lim_{\varepsilon \mapsto 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon} = \frac{d}{d\varepsilon} f(x + \varepsilon h)\Big|_{\varepsilon=0}.$$

According to Taylor's theorem,

$$f(x + \varepsilon h) = f(x) + \nabla f(x) \cdot (\varepsilon h) + O(\varepsilon^2).$$

Therefore,

$$D_h f(x) = \lim_{\varepsilon \mapsto 0} \frac{f(x+\varepsilon h) - f(x)}{\varepsilon} = \nabla f(x) \cdot h.$$

12. Let F be a vector field on $\mathbf{R^n}$. Thus $F : \mathbf{R^n} \mapsto \mathbf{R^n}$ by

$$F(x) = (F_1(x), \ldots, F_n(x)).$$

The divergence of the vector field F at x is

$$\operatorname{div} F(x) = \sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_i}.$$

The divergence measures the infinitesimal flux of the vector field. If $\operatorname{div} F(x)$ is positive, then the net flow of the vecotr field is *out* of the point x. Note that $\operatorname{div} F : \mathbf{R^n} \mapsto \mathbf{R}$. By the same token, if $f : \mathbf{R^n} \mapsto \mathbf{R}$, then ∇f is a vector field taking $\mathbf{R^n}$ to $\mathbf{R^n}$.

13. Let $f : \mathbf{R^n} \mapsto \mathbf{R}$ and $F : \mathbf{R^n} \mapsto \mathbf{R^n}$. Then $fF : \mathbf{R^n} \mapsto \mathbf{R^n}$ and

$$\operatorname{div}(fF) = \nabla f \cdot F + f \operatorname{div} F. \tag{1}$$

14. Let

$$dx = dx_1 \cdots dx_n,$$

be the n-dimensional volume differential. The integral of $f : \mathbf{R^n} \mapsto \mathbf{R}$ over $B \subseteq \mathbf{R^n}$ is denoted

$$\int_B f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \int_B f(x)\, dx.$$

15. Let Q be a smooth, closed surface in $\mathbf{R^n}$ enclosing a region B, with surface area differential dS. (So if $n = 3$, then Q really is a surface and dS the surface area differential. If $n = 2$, then Q is a closed curve and dS the arclength differential on Q.) Let $\nu = \nu(x)$ be the outer unit normal to Q at x. Let $F : \mathbf{R^n} \mapsto \mathbf{R^n}$ be a smooth vector field. The divergence theorem relates the integral over B to a surface integral over Q:

$$\int_Q F \cdot \nu\, dS = \int_B \operatorname{div} F\, dx.$$

16. Integration by parts: Let F be a vector field on $\mathbf{R^n}$ and $f : \mathbf{R^n} \mapsto \mathbf{R}$. Then

$$\begin{aligned}
\int_B f \operatorname{div} F\, dx &= \int_B \left[\operatorname{div}(fF) - F \cdot \nabla f\right]\, dx \\
&= \int_B \operatorname{div}(fF)\, dx - \int_B F \cdot \nabla f\, dx \\
&= \int_Q fF \cdot \nu\, dS - \int_B F \cdot \nabla f\, dx.
\end{aligned}$$