Queueing Theory 3: The Erlang Distribution

1. The convolution of the functions f and g is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy.$$

It is easy to show that convolution is commutative:

$$(f * g)(x) = (g * f)(x).$$

2. **Proposition:** If X and Y are independent random variables with probability density functions f and g, then $X + Y$ has density $f * g$.

3. Let $T_1, \ldots, T_n$ be independent, exponential random variables with parameter α. Thus, each of the T_i has probability density function

$$f_1(t) = \begin{cases} 0 & \text{for } t < 0, \\ \alpha e^{-\alpha t} & \text{for } t \geq 0. \end{cases}$$

By the previous proposition, $T_1 + T_2$ has density

$$\begin{aligned} (f * f)(t) &= \int_0^t f(t-s) f(s)\, ds \\ &= \begin{cases} 0 & \text{for } t < 0 \\ \alpha^2 e^{-\alpha t} & \text{for } t \geq 0. \end{cases} \\ &\equiv f_2(t) \end{aligned}$$

By the same token, $T_1 + T_2 + T_3$ has density function

$$\begin{aligned} (f * f * f)(t) &= (f * f_2)(t) \\ &= \begin{cases} 0 & \text{for } t < 0 \\ \frac{\alpha^3 t^2}{2} e^{-\alpha t} & \text{for } t \geq 0. \end{cases} \\ &\equiv f_3(t) \end{aligned}$$

By induction, we see that the density for $T_1 + \cdots + T_n$ is the n-fold convolution

$$\begin{aligned} \underbrace{(f * \cdots * f)}_{n \text{ times}}(t) &= (f * f_{n-1})(t) \\ &= \begin{cases} 0 & \text{for } t < 0 \\ \frac{\alpha^n t^{n-1}}{(n-1)!} e^{-\alpha t} & \text{for } t \geq 0. \end{cases} \\ &\equiv f_n(t) \end{aligned}$$

4. A random variable T with density function f_n is called *Erlang* distributed. When $n = 1$ the Erlang and exponential distributions coincide. Suppose that T is the sum of the independent, exponential random variables $T_1, \ldots, T_n$, each with parameter α. Then T is Erlang distributed with density f_n, and its mean is

$$E(T) = E(T_1) + \cdots + E(T_n) = \frac{n}{\alpha}.$$

Since the T_i are indepedent, the variance of T is

$$\text{Var}\,(T) = \text{Var}\,(T_1) + \cdots + \text{Var}\,(T_n) = \frac{n}{\alpha^2}.$$

5. **Example:** You join a queue with three people ahead of you. One is being served and two are waiting. Their service times S_1, S_2 and S_3 are independent, exponential random variables with common mean 2 minutes. (Thus the parameter is the mean service rate $\mu = .5$/minute.) Your conditional time in the queue *given* the system state $N = 3$ upon your arrival is

$$T = S_1 + S_2 + S_3.$$

T is Erlang distributed with density function f_3. The probability that you wait more than 5 minutes in the queue is

$$P[T > 5] = \frac{.5^3}{2} \int_5^\infty t^3 e^{-.5t}\, dt \approx .544.$$