

## Queueing Theory 2: The Poisson Process and the Exponential Distribution

1. A function  $f(h)$  is  $o(h)$  as  $h \rightarrow 0$  if

$$\lim_{h \rightarrow 0} \frac{f(h)}{h} = 0.$$

2. Let  $X(t)$  be the number of occurrences of some event in the time period  $(0, t]$ . We are interested in the distribution

$$P[X(t) = k], \quad k = 0, 1, 2, \dots$$

of  $X(t)$ . Note that the number of occurrences in  $(s, t]$  is  $X(t) - X(s)$ .

3. Suppose that:

a.  $X(0) = 0$ .

b.  $X(t)$  has independent increments. This means that, given times  $t_1 < t_2 < \dots < t_m$ , the random variables  $\{X(t_2) - X(t_1), \dots, X(t_m) - X(t_{m-1})\}$  are independent.

c.  $X(t)$  has stationary increments. This means that for all  $k, t_1, t_2$  and  $h$ ,

$$P[X(t_1 + h) - X(t_1) = k] = P[X(t_2 + h) - X(t_2) = k].$$

d.  $P[X(h) = 1] = \alpha h + o(h)$  for some  $\alpha > 0$ .

e.  $P[X(h) \geq 2] = o(h)$ .

Note that (d) and (e) together imply that

$$P[X(h) = 0] = 1 - \alpha h + o(h). \tag{1}$$

4. **Proposition:** If  $X(t)$  satisfies (a)-(e), then for  $t \geq 0$  and  $n = 0, 1, 2, \dots$

$$P[X(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t}. \tag{2}$$

**Proof:** The proof presented in class is standard. See *Introduction to Probability Models*, by Sheldon Ross.

5. The stochastic process  $X(t)$  is called the Poisson process.

6. **Proposition:** The expected value of  $X(t)$  is  $E[X(t)] = \alpha t$ . Thus  $\alpha$  is the mean occurrence rate, that is, the average number of occurrences per unit time.

7. A random variable  $T$  is exponentially distributed with if it has a probability density function of the form

$$f(t) = \begin{cases} \alpha e^{-\alpha t} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

for some  $\alpha > 0$ . It isn't hard to show that

$$E(T) = \frac{1}{\alpha} \quad \text{and} \quad \text{Var}(T) = \frac{1}{\alpha^2}. \quad (3)$$

**8. Proposition:** Suppose that occurrences of an event are governed by a Poisson process  $X(t)$  with mean occurrence rate  $\alpha$ . For  $i = 1, 2, 3, \dots$  let  $T_i$  be the time between the  $i$ th and  $(i + 1)$ st occurrences. Then the interoccurrence times  $T_i$  are independent, exponential random variables with common parameter  $\alpha$ .

9. The converse is also true: If the interoccurrence times  $T_i$  are independent, exponentially distributed random variables with common parameter  $\alpha$ , then the occurrence process  $X(t)$  is Poisson with mean occurrence rate  $\alpha$ .

10. Think of customers' arrivals as the "occurrences," and the  $T_i$  as interarrival times. By the previous two paragraphs, customers arrive according to a Poisson process with mean rate  $\lambda$  if and only if the interarrival times  $T_i$  are independent, exponential random variables with common parameter  $\lambda$ .

11. **Example:** Customers arrive according to a Poisson input process at mean rate 2/minute. Thus the probability that there be at most  $N$  arrivals in a period of length  $t$  is

$$P[X(t) \leq N] = e^{-2t} \sum_{n=0}^N \frac{(2t)^n}{n!}.$$

Since the input process is Poisson, the interarrival times are independent, exponentially distributed random variables with mean .5 minutes. Thus the probability that more than more than  $t$  minutes elapse between the arrivals of the fourth and fifth customers is

$$P[T_5 > t] = e^{-2t}.$$

12. **Proposition:** Suppose that there are  $m$  different kinds of events. Events of the  $i$ th type are governed by a Poisson process  $X_i(t)$  with mean occurrence rate  $\alpha_i$ . If the processes  $X_1(t), \dots, X_m(t)$  are independent, then the aggregate occurrence process

$$X(t) = X_1(t) + \dots + X_m(t),$$

is Poisson with mean occurrence rate  $\alpha = \alpha_1 + \dots + \alpha_m$ .

**13. Proposition:** Let occurrences of some event be governed by a Poisson process with mean rate  $\alpha$ . Let each occurrence be one of  $m$  types, and  $p_i$  be the probability that an occurrence be of the  $i$ th type, with

$$p_1 + \cdots + p_m = 1.$$

If  $X_i(t)$  is the number of occurrences of type  $i$  in a period of length  $t$ , then  $X_i(t)$  is a Poisson process with mean rate  $p_i\alpha$ . For example, suppose that customers arrive at an ATM according to a Poisson input process with mean rate .4/minute. If 75% of the customers are female, and 25% male, then the arrivals of female and male customers are governed by Poisson processes with mean rates .3/minute and .1/minute respectively.

**14.** Let  $T$  be an exponential random variable. Then

$$P[T > t + s | T > t] = P[T > s]. \quad (4)$$

In other words, exponential random variables are memoryless.

**15.** Let  $T_1, \dots, T_n$  be independent exponential random variables with common parameters  $\alpha_1, \dots, \alpha_n$  respectively. Then

$$T = \min \{T_1, \dots, T_n\},$$

is exponentially distributed with parameter  $\alpha_1 + \cdots + \alpha_n$ .