1. The ODE is separable. Rewrite it as $y \, dy = \frac{x \, dx}{1 + x^2}$, integrate, solve for $y$, etc.

2. The equation is linear. An integrating factor is $\rho(x) = x^3$.

3. Let $x(t)$ be the amount of salt in the tank at time $t$ minutes. The conservation equation for $x$ is

$$x' = 4 - \frac{x}{50 + t},$$

and the initial value, $x(0) = 20$. The ODE is linear, and the initial value problem can be solved for $x(t)$ by the method of integrating factors. Note that the tank is gaining 2 gallons of brine per minute, and will thus be full at time $t = 50$ minutes. The amount of salt in the tank at that moment is $x(50)$.

4a. Model the temperature change with an IVP for Newton’s law of cooling:

$$\begin{align*}
T' &= -k(T - 70), \\
T(0) &= 180.
\end{align*}$$

4b. You can solve the ODE easily by separation of variables or the method of integrating factors. Then plug in the initial value to get $T(t) = 70 + 110e^{-kt}$.

Now use the fact that $T(4) = 160$ to show that $k = \frac{1}{4} \ln \left( \frac{11}{9} \right)$.

5. The differential form of the ODE is exact.

6. Rewrite the ODE as $y' + \frac{2}{x}y = \frac{5}{x^2}y^3$, which is a Bernoulli equation with $n = 3$. Linearize it by setting $v = y^{-2}$.

7. Set $y' = p$. Then $y'' = p \frac{dp}{dy}$. You now have a first-order ODE with independent variable $y$ and dependent variable $p$:

$$yp \frac{dp}{dy} + p^2 = yp,$$

or

$$\frac{dp}{dy} + \frac{1}{y}p = 1.$$

This equation is linear, with integrating factor $\rho(y) = y$. Solve it and then use $y' = p$ to recover the solution $y$ to the original ODE.

8a. Separate variables, integrate, and use the initial condition to get the solution

$$y_1(x) = \exp \left[ \left( \frac{1}{3}x^2 - \frac{1}{3} \right)^{\frac{3}{2}} \right].$$

8b. Another solution is $y_2(x) = 1$. Since $f(x, y) = 2xy(\ln y)^{2/3}$ has partial derivative $f_y(x, y)$ that is discontinuous at $(1, 1)$, the initial value problem is not guaranteed a unique solution by the existence-uniqueness theorem.