Vector Fields and Line Integrals

1. Let C be a curve traced by the vector-valued function

$$\vec{r}\,(t) = \langle x(t), y(t), z(t)\rangle, \tag{1}$$

for $a \leq t \leq b$. The arclength differential on C is

$$ds = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, dt. \tag{2}$$

As we saw in class, the *line integral* of the *function* $g : \mathbf{R}^3 \mapsto \mathbf{R}$ over C can be expressed as integral with respect to t:

$$\int_C g(x, y, z)\, ds = \int_a^b g(x(t), y(t), z(t)) \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, dt. \tag{3}$$

2. Let $\vec{F} : \mathbf{R}^3 \mapsto \mathbf{V}_3$ by

$$\vec{F} = \langle M, N, P\rangle. \tag{4}$$

We call $\vec{F}$ *conservative* if there is a function $f : \mathbf{R}^3 \mapsto \mathbf{R}$ such that

$$\vec{F} = \nabla f.$$

The function f is a *potential* for $\vec{F}$. Note that if f is a potential for $\vec{F}$, then for any constant c, $f + c$ is also a potential for $\vec{F}$.

3. Let

$$\vec{r} = \langle x, y, z\rangle. \tag{5}$$

The inverse-square field

$$\vec{F}(x, y, z) = \frac{k}{\| \vec{r} \|^3}\, \vec{r}, \tag{6}$$

is conservative in any region (not containing the origin) with potential

$$f(x, y, z) = -\frac{k}{\| \vec{r} \|}. \tag{7}$$

4. The line integral of vector field: Let $\vec{F} : \mathbf{R}^3 \mapsto \mathbf{V}_3$ by

$$\vec{F} = \langle M, N, P\rangle. \tag{8}$$

We set

$$\vec{r} = \langle x, y, z\rangle, \tag{9}$$

so that

$$d\vec{r} = \langle dx, dy, dz \rangle. \tag{10}$$

We may thus write the line integral of $\vec{F}$ over the oriented curve C as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy + P dz. \tag{11}$$

If $\vec{r} = \vec{r}\,(t)$ is given by (1), then

$$\vec{F} = \vec{F}(x(t), y(t), z(t)), \tag{12}$$

and

$$d\vec{r} = \vec{r}\,'(t)\, dt. \tag{13}$$

We can thus express the line integral of $\vec{F}$ over C as an integral with respect to t:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt. \tag{14}$$

When we defined the line integral of a *function*, we were only concerned with the length ds of an infinitesimal section of C. When we defined the line integral of a *vector field,* we had to consider both the length and direction of the infinitesimal displacement $d\vec{r}$ along C. For this reason, the curve C in (11) and (14) must be oriented. If $-C$ is the same curve with the opposite orientation, then

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}. \tag{15}$$

5. The curl of a vector field: The curl of $\vec{F} = \langle M, N, P \rangle$ is

$$\text{curl}\, \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}. \tag{16}$$

In the case of a two-dimensionsal field

$$\vec{F}(x, y) = \langle M(x, y), N(x, y) \rangle,$$

(16) reduces to

$$\text{curl}\, \vec{F} = (N_x - M_y)\vec{k}. \tag{17}$$

Remember that the curl of a vector field is another vector field.

6. Physical interpretation of the curl: Let C_ε be a circle of radius ε centered at (x, y, z), lying in the plane orthogonal to the unit vector $\vec{n}$. The *circulation* of $\vec{F}$ around C_ε is the line integral of $\vec{F}$ over C_ε. As we showed in class,

$$\operatorname{curl}\vec{F}(x,y,z)\cdot\vec{n} = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\oint_{C_\varepsilon}\vec{F}\cdot d\vec{r}. \tag{18}$$

Thus $\operatorname{curl}\vec{F}(x,y,z)$ is the infinitesimal circulation of $\vec{F}$, per unit area, abut (x, y, z), normal to $\vec{n}$. (You don't have to use concentric circles to define the curl. Any family of piecewise smooth, closed curves normal to $\vec{n}$ that can be shrunk to (x, y, z) will do.)

7. Stokes' Theorem: Let $\mathcal{S}$ be an oriented surface with unit normal $\vec{n}$, bounded by the closed curve $\partial\mathcal{S}$, oriented by the right-hand rule. Let $\vec{F}$ be a C^1 vector field. Then

$$\oint_{\partial\mathcal{S}}\vec{F}\cdot d\vec{r} = \iint_{\mathcal{S}}\operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma. \tag{19}$$

Think of $\mathcal{S}$ as the union of very small, almost flat, roughly rectangular patches. Let (x, y, z) lie in one such patch. Let $\vec{n}$ be the unit normal to $\mathcal{S}$ at that point. Since the patch is nearly flat, we can take $\vec{n}$ to be the unit normal to the entire patch. By the interpretation of the curl given in paragraph (6), the circulation about (x, y, z) normal to $\vec{n}$ is

$$\operatorname{curl}\vec{F}(x,y,z)\cdot\vec{n}\,d\sigma. \tag{20}$$

We saw in class that when we "add up" (i.e. integrate) this quantity over $\mathcal{S}$, the circulation over an *internal* patch boundary is cancelled by circulation about the adjacent patches. This leaves only the circulation about the boundary $\partial\mathcal{S}$. Thus the conclusion (19).

8. Green's Theorem: Let $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ be a *two-dimensional,* C^1 vector field. Let $\mathcal{S}$ be a region in the plane bounded by the closed curve $\partial\mathcal{S}$. We orient $\mathcal{S}$ by taking $\vec{n} = \vec{k}$, and $\partial\mathcal{S}$ by the counterclockwise direction. In two dimensions,

$$(\operatorname{curl}\vec{F})\cdot\vec{n} = (N_x - M_y)\vec{k}\cdot\vec{k} = N_x - M_y, \tag{21}$$

$$d\sigma = dA, \tag{22}$$

and

$$\vec{F}\cdot d\vec{r} = M dx + N dy. \tag{23}$$

Thus, Stokes' theorem becomes

$$\oint_{\partial\mathcal{S}}\vec{F}\cdot d\vec{r} \equiv \oint_{\partial\mathcal{S}} M dx + N dy = \iint_{\mathcal{S}}(N_x - M_y)\,dA. \tag{24}$$

This is the conclusion of Green's theorem. Bear in mind that it is just the two-dimesional version of Stokes' theorem.

9. Let the vector field $\vec{F}$ be C^1 on some simply connected region D. The following are equivalent:

a. $\vec{F}$ is conservative on D.

b. $\nabla \times \vec{F} = \vec{0}$ on D. (The vector field $\vec{F}$ is *irrotational* on D.)

c. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D.

d. $\int_C \vec{F} \cdot d\vec{r}$ is path-independent on D.

10. If $\vec{F}$ is conservative with potential f, then

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A), \tag{25}$$

where A and B are respectively the initial and terminal points of C

11. General advice on doing line integrals of vector fields: Let C be a curve lying in a simply connected region on which $\vec{F}$ is C^1. Suppose that you are to evaluate

$$I = \int_C \vec{F} \cdot d\vec{r}. \tag{26}$$

a. Compute $\operatorname{curl} \vec{F}$.

b. If $\operatorname{curl} \vec{F} = \vec{0}$ and C is closed, then by (25), $I = 0$.

c. If $\operatorname{curl} \vec{F} = \vec{0}$ and C is not closed, find a potential f and use (25).

d. If $\operatorname{curl} \vec{F} \neq \vec{0}$, and C is closed and lies in the xy-plane, try Green's theorem. The double integral (24) might be easier to evaluate than your original line integral. If C does not lie in the xy-plane, you *might* be able to use Stokes' theorem to simplify your calculation, but this is doubtful. The surface integral on the right-hand side of (19) is usually more complicated than the line integral on the left.

e. If all else fails, parametrize C and then use (14).