Bisections and Edge-Disjoint 1-factors

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Joint work with Stephen G. Hartke

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3 Extending Bollobás-Scott
Degree Sequences

Let $G$ be a graph. If you take the degree of each vertex and put them into a sequence $\pi$, what you have is a graphic degree sequence. We say that $G$ realizes $\pi$. 
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Example

$\pi = (4, 3, 2, 2, 2, 1),$
Let $G$ be a graph. If you take the degree of each vertex and put them into a sequence $\pi$, what you have is a graphic degree sequence. We say that $G$ realizes $\pi$.

Example

$$\pi = (4, 3, 2, 2, 2, 1),$$
Potential \( k \)-factors

We say \( \pi \) potentially has a \( k \)-factor if some realization of \( \pi \) has a \( k \)-factor.
We say $\pi$ *potentially* has a $k$-factor if some realization of $\pi$ has a $k$-factor.
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Finding a 1-factor

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**Finding a 1-factor**

$\pi = (4, 3, 2, 2, 2, 1)$
Kundu’s $k$-factor Theorem

Theorem (Kundu 1973)

A graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ has a potential $k$-factor if and only if $\pi - k = (d_1 - k, d_2 - k, \ldots, d_n - k)$ is also graphic.
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$$\pi = (4, 3, 2, 2, 2, 1) \quad \pi - 1 = (3, 2, 1, 1, 1, 0)$$
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$$\pi = (4, 3, 2, 2, 2, 1) \quad \pi - 1 = (3, 2, 1, 1, 1, 0)$$

Graph $G$ with vertices 1, 2, 3, 4, 2, 1.
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$$\pi = (4, 3, 2, 2, 2, 1)$$

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Kundu’s $k$-factor Theorem

Remark
There is no potential $f$-factor theorem.
Assume $\pi$ has even length.
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Conjecture (Busch, Ferrara, Hartke, Jacobson, Kaul and West)

A graphic degree sequence $\pi$ potentially has $k$ edge-disjoint 1-factors if and only if $\pi - k$ is graphic.
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**Theorem (Busch et. al.)**

If graphic sequence $\pi$ has minimum degree $\frac{n}{2} + k - 2$, then $\pi$ has a realization with $k$ edge-disjoint 1-factors.
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- Find a large regular bipartite subgraph.
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Finding Large Bisections

A spanning balanced bipartite subgraph is called a bisection.

Conjecture (Bollobás-Scott 2002)
Every graph $G$ has a bisection $H$ such that for every vertex $v$

$$\deg_H(v) \geq \frac{1}{2} \deg_G(v).$$
Finding Large Bisections

Definition
A spanning balanced bipartite subgraph is called a bisection.
Finding Large Bisections

**Definition**

A spanning balanced bipartite subgraph is called a *bisection*.

**Conjecture (Bollobás-Scott 2002)**

*Every graph $G$ has a bisection $H$ such that for every vertex $v$*

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor.$$
Finding Large Bisections

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This is tight if true:
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Bisections and Edge-Disjoint 1-factors

Finding Large Bisections

Bollobás-Scott Conjecture

Finding Large Bisections

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Large Potential Bisections

Conjecture (Bollobás-Scott 1999)

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Large Potential Bisections

Conjecture (Bollobás-Scott 1999)

Every graph $G$ has a bisection $H$ such that for every vertex $v$

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor.$$ 

Theorem (Potential Version, Hartke, S)

Every graphic sequence $\pi$ has a realization $G$ with a bisection $H$ such that for every vertex $v$

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v) - 1}{2} \right\rfloor.$$
Potential Bollobás-Scott

Let $u = (d_1 \geq d_2 \geq \cdots \geq d_n)$. We use a strengthened form of Havel-Hakimi due to Kleitman and Wang.

Theorem (Kleitman, Wang) Fix any $i$. The sequence $u$ is graphic if and only if, the sequence $(d_1 - 1, d_2 - 1, \ldots, d_i - 1, d_i + 1, \ldots, d_n)$ is graphic.
Let $\pi = (d_1 \geq d_2 \geq \cdots \geq d_n)$. 
Potential Bollobás-Scott

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Bisections and Edge-Disjoint 1-factors

Finding Large Bisections

Potential Case

Potential Bollobás-Scott

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**Theorem (Kleitman, Wang)**

*Fix any \( i \). The sequence \( \pi \) is graphic if and only if, the sequence*

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(d_1 - 1, d_2 - 1, \ldots, d_d - 1, \ldots, d_n)
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Potential Bollobás-Scott

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- We use a strengthened form of Havel-Hakimi due to Kleitman and Wang.

**Theorem (Kleitman, Wang)**

*Fix any $i$. The sequence $\pi$ is graphic if and only if, the sequence*

$$(d_1 - 1, d_2 - 1, \ldots, d_{d_i} - 1, d_{d_i+1}, \ldots, d_{i+1}, d_{i-1}, \ldots, d_n)$$
Let \( \pi = (d_1 \geq d_2 \geq \cdots \geq d_n) \).

We use a strengthened form of Havel-Hakimi due to Kleitman and Wang.

**Theorem (Kleitman, Wang)**

*Fix any* \( i \). The sequence \( \pi \) is graphic if and only if, the sequence

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(d_1 - 1, d_2 - 1, \ldots, d_{d_i} - 1, d_{d_i+1}, \ldots, d_{i+1}, d_{i-1}, \ldots, d_n)
\]

*is graphic.*
Every graphic sequence $u = d_1 \geq \cdots \geq d_n$ has a realization $G$ with a bisection $H$ such that for every vertex $v$ $\deg_H(v) \geq \deg_G(v) - 1/2$.
Every graphic sequence $\pi$ has a realization $G$ with a bisection $H$ such that for every vertex $v$

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- Let $\pi = d_1 \geq \cdots \geq d_n$.  

Potential Bollobás-Scott, Proof Outline
Every graphic sequence $\pi$ has a realization $G$ with a bisection $H$ such that for every vertex $v$

$$\deg_H(v) \geq \left\lfloor \frac{\deg_G(v) - 1}{2} \right\rfloor.$$

Let $\pi = d_1 \geq \cdots \geq d_n$.

Partition as so:

- $d_1$ •
- $d_2$ •
- $d_3$ •
- $d_4$ •
- $d_5$ •
- $d_6$ •
- $d_6$ •
- $d_8$ •
Every graphic sequence $\pi$ has a realization $G$ with a bisection $H$ such that for every vertex $v$

$$\deg_H(v) \geq \left\lceil \frac{\deg_G(v) - 1}{2} \right\rceil.$$ 

- Let $\pi = d_1 \geq \cdots \geq d_n$.
- Partition as so:
- Use Kleitman-Wang to add adjacencies.
A few more details
A few more details
A few more details
A few more details
Application to Kundu Generalization
So we have a large potential bisection.
Application to Kundu Generalization

- So we have a large potential bisection.
- Can we get a large regular bisection from that?
Application to Kundu Generalization

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Theorem (Csaba 2007)

Every balanced bipartite graph, each part size $n$, of minimum degree $\delta$, with $\delta \geq n/2$, has a regular spanning subgraph of size at least

$$\frac{\delta + \sqrt{2\delta n - n^2}}{2}$$
Application to Kundu Generalization

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Theorem (Csaba 2007)

Every balanced bipartite graph, each part size $n$, of minimum degree $\delta$, with $\delta \geq n/2$, has a regular spanning subgraph of size at least

$$\delta + \sqrt{2\delta n - n^2} \over 2$$
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**Theorem (Csaba 2007)**

*Every balanced bipartite graph, each part size n, of minimum degree $\delta$, with $\delta \geq n/2$, has a regular spanning subgraph of size at least*

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$$\frac{2}{2}$$
So what do you get?

Theorem (Hartke, S)

Every graphic sequence of minimum degree $\delta$ at least $n/2 + 2$ has a realization with $\delta - 2 + p_n(2\delta - n - 4)$ edge-disjoint 1-factors.
So what do you get?

Theorem (Hartke, S)

Every graphic sequence of minimum degree $\delta$ at least $n/2 + 2$ has a realization with

$$\delta - 2 + \frac{\sqrt{n(2\delta - n - 4)}}{2}$$

edge-disjoint 1-factors.
Approximate Bollobás-Scott

We showed the following approximate version of Bollobás-Scott (Independently, Albert Bush has a similar result):

Theorem

Every graph $G$ has a bisection $H$ where every vertex $v$ satisfies

$$\deg_H(v) \geq \deg_G(v) \left( 2 - p \ln n \right)$$
We showed the following approximate version of Bollobás-Scott (Independently, Albert Bush has a similar result)
Approximate Bollobás-Scott

We showed the following approximate version of Bollobás-Scott (Independently, Albert Bush has a similar result)

**Theorem**

*Every graph $G$ has a bisection $H$ where every vertex $v$ satisfies*

$$\deg_H(v) \geq \frac{\deg_G(v)}{2} - \sqrt{\deg(v) \ln n}$$
Approximate Bollobás-Scott

\[ \deg_H(v) \geq \frac{1}{2} \deg_G(v) - \sqrt{\deg_G(v) \ln(n)}. \]

Proof Outline.
Approximate Bollobás-Scott

\[ \deg_H(v) \geq \frac{1}{2} \deg_G(v) - \sqrt{\deg_G(v) \ln(n)}. \]

**Proof Outline.**

- Arbitrarily pair the vertices.
Approximate Bollobás-Scott

\[ \deg_H(v) \geq \frac{1}{2} \deg_G(v) - \sqrt{\deg_G(v) \ln(n)}. \]

**Proof Outline.**

- Arbitrarily pair the vertices.
- Randomly split each pair between the two partite sets.
Approximate Bollobás-Scott

\[ \deg_H(v) \geq \frac{1}{2} \deg_G(v) - \sqrt{\deg_G(v) \ln(n)}. \]

**Proof Outline.**

- Arbitrarily pair the vertices.
- Randomly split each pair between the two partite sets.
- Bound the probability of a bad vertex using Chernoff bounds.
Approximate Bollbás-Scott

\[ \deg_H(v) \geq \frac{1}{2} \deg_G(v) - \sqrt{\deg_G(v) \ln(n)}. \]

**Proof Outline.**

- Arbitrarily pair the vertices.
- Randomly split each pair between the two partite sets.
- Bound the probability of a bad vertex using Chernoff bounds.
- Combine using the union sum bound.
Approximate Bollobás-Scott

\[
\deg_H(v) \geq \frac{1}{2} \deg_G(v) - \sqrt{\deg_G(v) \ln(3\Delta)}.
\]

**Proof Outline.**

- Arbitrarily pair the vertices.
- Randomly split each pair between the two partite sets.
- Bound the probability of a bad vertex using Chernoff bounds.
- Combine using the Lovász Local Lemma.
Combining the bisection result with Csaba’s Theorem on finding regular bipartite subgraphs, we get
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**Theorem (Hartke, S)**

*Any graph of minimum degree $n/2 + \sqrt{2n \ln n}$ has at least $n/8$ edge-disjoint 1-factors.*
By extending the same probabilistic argument, we have
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**Theorem (Hartke, S)**

Let $G$ be a graph on $n$ vertices, where $n = pq$ for $p > 1$. Then there exists a partition of the vertices of $G$ into $q$ parts of size $p$ such that every vertex $v$ has at least $\frac{\deg(v)}{q} - \sqrt{\deg(v) \cdot \ln(n)}$ neighbors in each part.
Example
Example
Example
Example
Any graph with $n$ vertices, $n$ and even square, of minimum degree $n/2 + (n \ln n)^{3/4}$ has $n/4 - \sqrt{n}/4$ edge disjoint 1-factors.
Thanks!