Work all of the following problems, and turn in a set of solutions as a group.

**Theorem 2.1**  Let $G$ be a non-abelian torsion group and $K$ be a field. If a group ring $K[G]$ is reversible, then $G$ is hamiltonian.

1. Complete the details of the proof of Theorem 2.1 using the following outline:
   - Assume that $x \in G \setminus \{1\}$, and let $n$ denote the order of $x$.
   - Show that for all $y \in G$ we have
     $$y(1-x)(1+x+\cdots+x^{n-1}) = 0.$$  
   - Using the fact that $K[G]$ is reversible, show that
     $$(1+x+\cdots+x^{n-1})(1-xyy^{-1}) = 0.$$  
   - Show that $\{1,x,\ldots,x^{n-1}\}$ is a set of $n$ pairwise distinct elements of $G$, and use this fact to show that
     $$\{1,x,\ldots,x^{n-1}\} = \{1,x,\ldots,x^{n-1}\} yxy^{-1}.$$  
   - Conclude that $yxy^{-1} \in \langle x \rangle$, and hence $\langle x \rangle \unlhd G$.
   - This shows that every cyclic subgroup of $G$ is normal; extend this argument to an arbitrary subgroup $H$ of $G$. Hint: Write $H = \langle x_1, x_2, \ldots \rangle$.

2. Show that the group of Quaternions is a Hamiltonian group. (Hint: You may want to recall that if $G$ is a finite group of even order, and $H < G$ such that $[G: H] = 2$, then $H \unlhd G$.)

3. Prove the following:
   (a) That any finite group has finite exponent.
   (b) Give an example of an infinite group with finite exponent.
   (c) Does a finite group of exponent $m$ always contain an element of order $m$? Explain.

4. Let $G$ be a finite abelian group. Prove
   $$\exp(G) = \text{lcm}\{\circ(g) \mid g \in G\}$$
   (a) Is finite necessary?
   (b) Is abelian necessary?

5. Show that $K[G \times H] \cong K[G][H]$ for every field $K$ and groups $G$ and $H$. 