Definition. Let $R$ be a ring with identity. Define $P_n(R)$ to be the set $\sigma$ such that $x_1 \cdots x_n = 0$ implies $x_{\sigma(1)} \cdots x_{\sigma(n)} = 0$.

Note. (1) $I_n$ is the trivial permutation on $n$ elements
(2) $\Sigma_n$ is the set of all permutations on $n$ elements
(3) $C_n$ is the subgroup generated by the $n$--cycle $(12 \cdots n)$

Question. What permutation is always in $P_n(R)$?

Lemma. The following properties hold for permutations:
(1) If $\sigma \neq I_n$ then there exists $j < k$ with $\sigma(j) > \sigma(k)$.
(2) $\Sigma_n$ can be generated by any transposition and the $n$--cycle $(12 \cdots n)$.
(3) For all $1 \leq j \leq n$, show there exists some $\sigma \in C_n$ such that $\sigma(j) = 1$.
(4) The set $P_n(R)$ is a subgroup of $\Sigma_n$.

Proof. Exercise 1.

Theorem. Let $R$ be a ring with identity. Then one of the following hold:
(1) $P_n(R) = I_n$ for all $n \geq 2$
(2) $P_n(R) = C_n$ for all $n \geq 2$
(3) $P_n(R) = \Sigma_n$ for all $n \geq 2$

Proof. First note that either $P_2(R) = I_2$ or $P_2(R) = C_2 = \Sigma_2$. (Exercise 2)

Case 1. $P_2(R) = I_2$, i.e., there exists $a, b \in R$ with $ab = 0$ but $ba \neq 0$.

Let $n \geq 2$ be given. We wish to show $P_n(R) = I_n$. Since $I_n \in P_n(R)$, it’s enough to show if $\sigma \neq I_n$ then $\sigma \notin P_n(R)$.

So let $\sigma \neq I_n$.

Exercise 3. Find $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n = 0$ but $x_{\sigma(1)} \cdots x_{\sigma(n)} \neq 0$ and conclude $P_n(R) = I_n$.

[HINT: Use property 1 of the lemma.]

Case 2. $P_2(R) = C_2 = \Sigma_2$, i.e., $ab = 0$ implies $ba = 0$ for all $a, b \in R$.

Let $n \geq 2$ be given.

Exercise 4. Show the permutation $(12 \cdots n) \in P_n(R)$ and conclude $C_n \subseteq P_n(R)$. [HINT: Use property 4 of the lemma.]

We now wish to show that if $C_n \subseteq P_n(R)$ for some $n$, then $P_n(R) = \Sigma_n$ for all $n \geq 2$. To do so, we must first show $R$ is symmetric. Choose a permutation $\delta \in P_n(R) \setminus C_n$. Say $\delta(1) = j$.

Exercise 5. Using the permutation $\delta$ and property 3 of the lemma, show there exists a permutation $\sigma \notin I_n$ with $\sigma(1) = 1$.

Exercise 6. Use Exercise 5 and property 1 of the lemma to conclude $R$ is symmetric.

In order to show $P_n(R) = \Sigma_n$ for all $n \geq 2$, note by properties 2 and 4 of the lemma, it is enough to show $P_n(R)$ contains the permutations $(12 \cdots n)$ and $(12)$. As we’ve shown $C_n \subseteq P_n(R)$, it contains $(12 \cdots n)$.

Exercise 7. Let $x_1 \cdots x_n = 0$ for $x_1, \ldots, x_n \in R$. Use exercise 6 to conclude $x_2 x_1 x_3 \cdots x_n = 0$ and conclude $P_n(R) = \Sigma_n$.

Corollary. Let $R$ be a ring with identity. Then
(1) $P_n(R) = I_n$ if and only if $R$ is not reversible nor symmetric.
(2) $P_n(R) = C_n$ if and only if $R$ is reversible, but not symmetric.
(3) $P_n(R) = \Sigma_n$ if and only if $R$ is reversible and symmetric.

Proof. Exercise 8.