IMMERSE 2010
Algebra Course
Problem Set 9

Work all of the following problems, and turn in a set of solutions as a ring. Each group member should be responsible for \TeX-ing at least one problem.

1. Let $\phi$ be a ring homomorphism from $R$ to $S$. Prove that $R/\ker(\phi)$ is isomorphic to $\phi(R)$.

2. Recall a ring element is called idempotent if $a^2 = a$. Prove that a ring homomorphism carries an idempotent to an idempotent.

3. Suppose that $R$ and $S$ are commutative rings with unities. Let $\phi$ be a ring homomorphism from $R$ to $S$ and let $A$ be an ideal of $S$.

   (a) If $A$ is prime in $S$, show that $\phi^{-1}(A) = \{x \in R \mid \phi(x) \in A\}$ is prime in $R$.

   (b) If $A$ is maximal in $S$, show that $\phi^{-1}(A)$ is maximal in $R$.

4. Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{Z} \right\}$, and let $\phi$ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to $a - b$.

   (a) Show that $\phi$ is a homomorphism.

   (b) Determine the kernel of $\phi$.

   (c) Show that $R/\ker(\phi)$ is isomorphic to $\mathbb{Z}$.

   (d) Is $\ker(\phi)$ a prime ideal?

   (e) Is $\ker(\phi)$ a maximal ideal?