IMMERSE 2010
Algebra Course
Problem Set 3

Work all of the following problems, and turn in a set of solutions as a group. Each group member should be responsible for \TeX-ing two problems. The final two problems are ‘Challenge problems’ that we encourage you to work through, and turn in your best effort.

1. Let $G$ be a group and $g \in G$. Prove that
   \[ \langle x \rangle := \{ x^n | n \in \mathbb{Z}^+ \} \]
   is a subgroup of $G$.

2. For $G$ a group, the center of $G$ is the set
   \[ Z(G) := \{ a \in G | ga = ag \ \forall g \in G \} \]
   Prove that $Z(G)$ is a subgroup of $G$.

3. Let $H, K$ be subgroups of a group $G$. Prove that
   \( (a) \ H \cap K \leq G. \)
   \( (b) \ H \cup K \leq G \) if and only if $H \leq K$ or $K \leq H$.

4. Prove that every subgroup of a cyclic group is cyclic.

5. Describe all subgroups of $\mathbb{Z}_{24}$. Specifically, give a generator for each subgroup, find the order of the subgroup, and describe the containments among subgroups.

6. Describe all subgroups of $S_3$.

7. Let $G$ be a group, $H$ a subgroup of $G$ and $g \in G$ be fixed. Show
   \[ gHg^{-1} := \{ ghg^{-1} | h \in H \} \]
   is a subgroup of $G$, with $|H| = |gHg^{-1}|$

8. Let $G = \langle x \rangle$ be a finite group order $n$. Show $x^a = x^b$ if and only if $a \equiv b \pmod{n}$.


10. Challenge problem! How many elements of a cyclic group of order $n$ are generators for that group?