1. Let $\sigma = \left( \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{array} \right)$ and $\tau = \left( \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{array} \right)$. Write each of the following as a product of disjoint cycles.

Solution (by Kirsten, Nathan M, Julia, Derrek):

(a) $\sigma = (135)(24)$

(b) $\tau = (15)(23)$

(c) $\sigma^2 = (153)$

(d) $\sigma\tau = (2534)$

(e) $\tau\sigma = (1243)$

(f) $\tau^2\sigma = (135)(24)$

2. Find the product on $S_6$.

Solution (by Nathan S, Jagya, Tim, Lilith)

(a) $(13)(15426)(2615) = (1423)(56)$

(b) $(13)(42)(12)(14)(23)(23) = (123)$

(c) $(13)(42)(12)(14) = (123)$

3. List the symmetries of the letter $H$ and give the table of this group of symmetries.

Solution (by Kirsten, Nathan M, Julia, Derrek) The symmetries are: the identity ($R_0$), rotation by 180 degrees ($R_{180}$), reflection across the horizontal axis ($H$), and reflection across the vertical axis ($V$).

The table for these symmetries is as follows:

<table>
<thead>
<tr>
<th></th>
<th>$R_0$</th>
<th>$R_{180}$</th>
<th>$H$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0$</td>
<td>$R_0$</td>
<td>$R_{180}$</td>
<td>$H$</td>
<td>$V$</td>
</tr>
<tr>
<td>$R_{180}$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
<td>$V$</td>
<td>$H$</td>
</tr>
<tr>
<td>$H$</td>
<td>$H$</td>
<td>$V$</td>
<td>$R_0$</td>
<td>$R_{180}$</td>
</tr>
<tr>
<td>$V$</td>
<td>$V$</td>
<td>$H$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
</tr>
</tbody>
</table>

4. Show that the inverse of $(a_1a_2\ldots a_k)$ in $S_n$ is $(a_1a_ka_{k-1}\ldots a_2)$.

Proof (by Nathan S, Jagya, Tim, Lilith). Let $\alpha = (a_1a_2\cdots a_k)$ and $\beta = (a_1a_ka_{k-1}\cdots a_2)$.

We observe that $\beta(\alpha(a_k)) = \beta(a_1) = a_k$. Also for $1 \leq i < k$, $\beta(\alpha(a_i)) = \beta(a_{i+1}) = a_i$. Thus $\beta\alpha$ is the identity.

Observing $\alpha\beta$ we see that $\alpha(\beta(a_1)) = \alpha(a_k) = a_1$. Also for $1 < i \leq k$, $\alpha(\beta(a_i)) = \alpha(a_{i-1}) = a_i$. Thus $\alpha\beta$ is the identity. Thus the inverse of $\alpha$ is $\beta$.

5. Let $\tau$ be a transposition and let $\sigma \in S_n$. Prove that $\sigma\tau\sigma^{-1}$ is a transposition.

Proof (by Caitlyn, Chris, Sarah, Linda). Let $\sigma = (x_1x_2\ldots x_n)$ and $\tau = (ab)$. Then we know that $\sigma^{-1} = (x_n x_{n-1} \ldots x_1)$. There are three cases.

Case 1: Let $a, b \in \{x_1, x_2, \ldots, x_n\}$, so $a = x_j$ and $b = x_k$ for some $1 \leq j, k \leq n$ where $j \neq k$. Consider $x_{j+1}$ and $x_{k+1}$. We know that $x_{j+1} \mapsto x_j \mapsto x_k \mapsto x_{k+1}$, so $x_{j+1} \mapsto x_{k+1}$ and $x_{k+1} \mapsto x_k \mapsto x_{j+1}$, so $x_{k+1} \mapsto x_{j+1}$. Now consider $x_i \neq x_{j+1}, x_{k+1}$. We know that $x_i \mapsto x_{i-1} \mapsto x_i$, so everything except
8. Define \(x_j+1\) and \(x_{k+1}\) maps to itself. Therefore, \(\sigma \tau^{-1} = (x_1x_2...x_n)(ab)(x_{n}x_{n-1}...x_1) = (x_{k+1}x_{j+1})\), which is a transposition, as was to be shown.

Case 2: Let \(a, b \notin \{x_1, x_2, ..., x_n\}\). So \(x_i \mapsto x_{i-1} \mapsto x_i\) for all \(1 \leq i \leq n\). We also know that \(a \mapsto b\) and \(b \mapsto a\). Therefore \(\sigma \tau^{-1} = (x_1x_2...x_n)(ab)(x_{n}x_{n-1}...x_1) = (ab)\), which is a transposition, as was to be shown.

Case 3: Let \(a \in \{x_1, x_2, ..., x_n\}\) and \(b \notin \{x_1, x_2, ..., x_n\}\), so \(a = \sigma_k\) for some \(1 \leq k \leq n\). Consider \(x_{k+1}\). We know that \(x_{k+1} \mapsto x_k \mapsto b\), so \(x_{k+1} \mapsto x_k \mapsto x_{k+1}\). Then for \(x_i \neq x_{k+1}\), we have \(x_i \mapsto x_{i-1} \mapsto x_i\), so everything except \(x_{k+1}\) and \(b\) maps to itself. Therefore, \(\sigma \tau^{-1} = (x_1x_2...x_n)(ab)(x_{n}x_{n-1}...x_1) = (x_{k+1}b)\), which is a transposition, as was to be shown. Similarly, for \(a \notin \{x_1, x_2, ..., x_n\}\) and \(b \in \{x_1, x_2, ..., x_n\}\) with \(b = \sigma_k\) for some \(1 \leq k \leq n\), we have \(\sigma \tau^{-1} = (x_1x_2...x_n)(ab)(x_{n}x_{n-1}...x_1) = (x_{k+1}a)\).

Thus, \(\sigma \tau^{-1}\) is a transposition.

6. Let \(\sigma\) and \(\tau\) be disjoint cycles in \(S_n\). Prove that \(\sigma \tau = \tau \sigma\).

**Proof (by Caitlyn, Chris, Sarah, Linda).** Let \(\sigma = (\sigma_1\sigma_2...\sigma_n)\) and \(\tau = (\tau_1\tau_2...\tau_m)\). Then \(\sigma \tau = (\sigma_1\sigma_2...\sigma_n)(\tau_1\tau_2...\tau_m)\).

Because the cycles are disjoint, we see that this permutation carries \(\sigma_1\) to \(\sigma_2\), \(\sigma_2\) to \(\sigma_3\), and so forth down to \(\sigma_n\) going to \(\sigma_1\). Similarly, the permutation takes \(\tau_1\) to \(\tau_2\), \(\tau_2\) to \(\tau_3\), and so on until \(\tau_m\) is taken to \(\tau_1\). Now consider the product \(\sigma \tau = (\tau_1\tau_2...\tau_m)(\sigma_1\sigma_2...\sigma_n)\). Again noting what the permutation does to individual elements, we see that \(\sigma_1\) goes to \(\sigma_2\), \(\sigma_2\) goes to \(\sigma_3\), and so forth down to \(\sigma_n\) going to \(\sigma_1\). Also, it takes \(\tau_1\) to \(\tau_2\), \(\tau_2\) to \(\tau_3\), and so on until \(\tau_m\) goes to \(\tau_1\). Since both products take each element to the same place, they are the same permutation. Thus, disjoint cycles commute.

7. Let \(\sigma\) and \(\tau\) be disjoint cycles in \(S_n\). Prove for all \(i\), that \(\sigma\) moves \(i\) if and only if \(\sigma^{-1}\) moves \(i\).

**Proof (by Anne, Annette, Lisa, and Matthew).** (\(\Rightarrow\)) Suppose \(\sigma\) moves \(i\) and \(\sigma^{-1}\) does not. Then \(\sigma(i) = j\) and \(\sigma^{-1}(i) \neq i\). Consider the composition \(\sigma(\sigma^{-1}(i)) = \sigma(i) = j\), but then \(\sigma^{-1}\) and \(\sigma\) are not inverses, and we have a contradiction. Thus, \(\sigma^{-1}\) must move \(i\) as well.

(\(\Leftarrow\)) Suppose \(\sigma^{-1}\) moves \(i\) and \(\sigma\) does not. Then \(\sigma^{-1}(i) = k\) and \(\sigma(i) \neq i\). Consider the composition \(\sigma^{-1}(\sigma(i)) = \sigma^{-1}(i) = k\), but then \(\sigma\) and \(\sigma^{-1}\) are not inverses, and we have a contradiction. Thus, \(\sigma\) must move \(i\) as well.

8. Define \(f: \{0, 1, 2, ..., 10\} \rightarrow \{0, 1, 2, ..., 10\}\) by \(f(n) = \text{the remainder after dividing by } 4n^2 - 3n^7\) by 11.

(a) Show that \(f\) is a permutation.

(b) Compute the parity of \(f\).

(c) Compute the inverse of \(f\).

**Proof (by Caitlyn, Chris, Sarah, Linda).** (a) Show that \(f\) is a permutation.

Consider the following cases, where \(n \in \mathbb{Z}_{11}\):

- If \(n = 0\), \(4n^2 - 3n^7 \pmod{11} = 0\)
- If \(n = 1\), \(4n^2 - 3n^7 \pmod{11} = 1\)
- If \(n = 2\), \(4n^2 - 3n^7 \pmod{11} = 6\)
- If \(n = 3\), \(4n^2 - 3n^7 \pmod{11} = 9\)
- If \(n = 4\), \(4n^2 - 3n^7 \pmod{11} = 5\)
- If \(n = 5\), \(4n^2 - 3n^7 \pmod{11} = 3\)
- If \(n = 6\), \(4n^2 - 3n^7 \pmod{11} = 10\)
- If \(n = 7\), \(4n^2 - 3n^7 \pmod{11} = 2\)
- If \(n = 8\), \(4n^2 - 3n^7 \pmod{11} = 8\)
If $n = 9$, $4n^2 - 3n^7 \pmod{11} = 4$
If $n = 10$, $4n^2 - 3n^7 \pmod{11} = 7$.
Thus $f$ is the permutation $(2 \ 6 \ 10 \ 7)(3 \ 9 \ 4 \ 5)$.

(b) Compute the parity of $f$.
The parity of $f$ can be computed by decomposing the permutation into a product of 2-cycles, resulting in
$$f = (2 \ 6 \ 10 \ 7)(3 \ 9 \ 4 \ 5) = (2 \ 7)(2 \ 10)(2 \ 6)(3 \ 5)(3 \ 4)(3 \ 9)$$
Since $f$ can be written as the product of 6 2-cycles, it is an even permutation.

(c) Compute the inverse of $f$.
The inverse of a permutation is the permutation which contains each element in the opposite order. Thus the inverse of $(2 \ 6 \ 10 \ 7)(3 \ 9 \ 4 \ 5)$ is $(7 \ 10 \ 6 \ 2)(5 \ 4 \ 9 \ 3)$. Note that $(7 \ 10 \ 6 \ 2)(5 \ 4 \ 9 \ 3) = (5 \ 4 \ 9 \ 3)(7 \ 10 \ 6 \ 2)$ since the permutations are disjoint by question (6).