Work all of the following problems, and turn in a set of solutions as a field. Each group member should be responsible for $\LaTeX$-ing at least one problem.

1. Find the following:
   (a) $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}]$.
   (b) $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$.

2. Determine all subfields of the following:
   (a) $\mathbb{Q}(\sqrt{2})$.
   (b) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

3. Let $K/F$ be a field extension and let $\alpha \in K$. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

4. Give an example of a field $F$ and elements $a, b \in F$ such that $F(a, b) \neq F(a)$, $F(a, b) \neq F(b)$, and $[F(a, b) : F] < [F(a) : F][F(b) : F]$.

5. Let $f(x) = x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$. Set 
   
   $\alpha = \sqrt{1 + \sqrt{3}}$ and $\beta = \sqrt{1 - \sqrt{3}}$.

   (a) Prove that $f(x)$ is irreducible, and find the roots of $f(x)$.
   (b) Let $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$. Prove that $K \neq L$, and $K \cap L = \mathbb{Q}(\sqrt{3})$.

6. Let $F$ be a field and $f(x) \in F[x]$ an irreducible polynomial of degree $n$. Let $g(x) \in F[x]$ be a nonconstant polynomial, and set $h(x) = f(g(x))$. Let $q(x) \in F[x]$ be an irreducible factor of $h(x)$. Prove that $n$ divides the degree of $q(x)$. 