Finite time blow-up in nonlinear suspension bridge models

Petronela Radu, Daniel Toundykov, Jeremy Trageser *

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE, USA

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Abstract

This paper settles a conjecture by Gazzola and Pavani [10] regarding solutions to the fourth order ODE $w^{(4)} + kw''' + f(w) = 0$ which arises in models of traveling waves in suspension bridges when $k > 0$. Under suitable assumptions on the nonlinearity $f$ and initial data, we demonstrate blow-up in finite time. The case $k \leq 0$ was first investigated by Gazzola et al., and it is also handled here with a proof that requires less differentiability on $f$. Our approach is inspired by Gazzola et al. and exhibits the oscillatory mechanism underlying the finite-time blow-up. This blow-up is nonmonotone, with solutions oscillating to higher amplitudes over shrinking time intervals. In the context of bridge dynamics this phenomenon appears to be a consequence of mutually-amplifying interactions between vertical displacements and torsional oscillations.

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1. Introduction

The topic of suspension bridges is a celebrated area of applied mathematics filled with engineering marvels, as well as many dramatic events. One of the most notorious disasters is the Tacoma Narrows Bridge collapse of 1940. The collapse of the bridge had been previously explained by a resonance-like effect produced by a wind of under 80 km/h [1]; however, in recent
literature (e.g. [10,13]) it has been demonstrated that resonance theory does not accurately describe these vibrational patterns. The phenomenon of self-amplifying oscillations in bridge dynamics is now recognized to be far more complex than originally believed; see the wonderful historical overview of existing theories in [7,13]. To explain these dynamics, several models have been proposed. The following model based on the Euler–Bernoulli beam equation was introduced by Lazer and McKenna in [14] and investigated further in [13]:

$$u_{tt} + u_{xxxx} + \gamma u^+ = W(x,t), \quad x \in (0,L), \; t > 0. \quad (1)$$

In the above model $u$ denotes vertical displacement, $L > 0$ is the length of the bridge, $u^+ = \max\{u,0\}$, $\gamma u^+$ represents the force from the cables treated as springs with a one-sided restoring force, and $W$ accounts for additional forces such as weight and wind. In [15] McKenna and Walter investigated traveling waves for this model. After some normalization, traveling wave solutions to (1) necessarily satisfy

$$w''''(t) + kw'''(t) + f(w(t)) = 0 \quad (2)$$

with $f(s) = (s + 1)^+ - 1$. In [4], a smooth analog of this nonlinearity given by $f(t) = e^t - 1$ was considered.

Recent research (e.g. [7]) shows that the stability of a bridge can be critically affected by torsional oscillations, in particular, their interactions with vertical displacements. Consequently, a one-dimensional model in the above interpretation does not accurately describe large oscillations, as twisting effects would not be taken into account. Indeed, traveling wave solutions corresponding to (2) are global when $f(s) \in \text{Lip}_{\text{loc}}(\mathbb{R})$, $f(s)s > 0$ for $s \neq 0$, and $f(s)$ has at most linear growth either as $s \to +\infty$ or as $s \to -\infty$, as it was shown in [3].

Observations of actual bridge oscillations (Millennium Bridge [2]) and collapses (Tacoma Narrows Bridge [6]) reinforce the idea that torsional and vertical oscillations in suspension bridges are coupled. To model this interaction, Drábek et al. [5] introduced a second unknown function to measure potentially unbounded torsional effects. Subsequently, it was suggested in [9] that the coupling mechanism be incorporated into a one-dimensional model by allowing the forcing term $f$ to take arbitrarily large negative values. In this new model positive values of $w$ correspond to vertical oscillations, while negative ones describe torsional deformations. A suitable function $f$ would necessarily be sign preserving, e.g. $f(s) = s^3 + s$. For an extensive overview of the theory for ODEs of the form (2) see the book [16] by Peletier and Troy.

The study of traveling waves for (1) when vertical and torsional oscillations are unbounded has been a challenging open problem. The current paper investigates finite time blow-up of solutions to Eq. (2) when $f$ is a locally Lipschitz function unbounded as $|t| \to \infty$.

In their paper [10] Gazzola and Pavani offered an innovative proof of blow-up for the case $k \leq 0$; this range for $k$ corresponds to models of beams in tension where $-k \geq 0$ represents the tension [11]. The scenario $k > 0$ corresponds to traveling wave solutions

$$u(t,x) = w(x + ct)$$

of the Euler–Bernoulli equation, with $k = c^2$ where $c$ is the speed of the wave. Numerical evidence strongly supports the blow-up for $k > 0$, as presented in [10]. This case, however, remains open until now since positive values of the parameter $k$ critically alter some intrinsic invariants associated with the ODE. We settle the blow-up conjecture when
• $k \in (0, 2)$ for a large class of nonlinear functions; see Theorem 1.
• $k \geq 2$ for scaled versions of nonlinearities satisfying the hypothesis of Theorem 1; see Coro-
lary 1.

Moreover, we cover the case $k \leq 0$ with an alternate proof that requires less regularity on $f$
and $w$. For all ranges of $k$, the assumptions on $f$ are satisfied for power-type nonlinearities. We
mention that the splitting into the cases $k < 2$ and $k \geq 2$ happens for technical reasons in our
proof; however, it may be related to the fact that the corresponding linearized system for (2) has
2-dimensional stable and unstable manifolds for $|k| < 2$ and has purely imaginary eigenvalues
when $k \geq 2$, see [3, Prop. 20].

The approach for all cases is inspired by the remarkable strategy developed in [10]; however,
major challenges had to be overcome to accommodate $k > 0$:

• First, most of the energy functionals used in [10] are not convex for $k > 0$, yet this ingredient
is critical in understanding the behavior of solutions. We introduce new energy functionals
adapted to (2) and take advantage of their convexity and monotonicity properties to describe
the behavior of solutions.
• An essential feature in the proof of the blow-up in [10] for $k \leq 0$ was the ability to ensure
the existence of exactly one inflection point between consecutive zeros of the function. The
same analysis does not extend to $k > 0$ so our proof allows multiple inflection points on an
interval of one sign. Numerical evidence seems to indicate that for sufficiently large energy
there is eventually only one inflection point between neighboring zeros; however, verifying
this conjecture is still an open problem.
• The intricate employment of test functions from [10] could not be reproduced for $k > 0$.
Instead, we rely on geometric features of energy functions to show the growth and blow-up
of solutions. This approach also allows us to handle less regular solutions, since we do not
need to differentiate the ODE multiple times.

The theoretical work described above corroborates preliminary numerical computations that we
have performed; together they prompt the following remarks with physical implications:

• For fourth order ODEs, the blow-up phenomenon, although oscillatory, seems to be driven
not by frequency (as in resonance for second-order ODEs), but rather by the amplitude of
the oscillations. Nonmonotone blow-up in finite time cannot be reproduced with a time-
dependent external forcing that matches the frequency of the system. Instead, the mechanism
is based on a transversal displacement inducing a torsional oscillation and vice versa. This
“dual-excitation” process leads to a finite-time blow-up of the traveling wave solution.
• The forcing term $f(u)$ considered in this paper is defocusing from the perspective of the
hyperbolic or Petrovsky dynamics. However, in the context of traveling wave formation this
restoring force has the opposite effect and induces a locally unbounded profile.

Our results are applicable to the study of some partial differential equations, e.g. Zakharov–
Kuznetsov equations [12] and biharmonic coercive equations [8, Section 3].

The paper is organized as follows. In Section 2 we present our main result, Theorem 1, regard-
ing Eq. (2): a sufficient condition for finite time blow-up. Section 2 also includes remarks and
a few open problems. Section 3 presents a comprehensive list of constants and energy functions
used throughout the paper. In Section 4 we prove several lemmas about the energies introduced
in Section 3, while Section 5 contains some results concerning the shape of the graphs of solutions to Eq. (2). In Section 6 we derive estimates for solutions to (2) and complete the proof of Theorem 1.

2. Main blow-up result

Assume that \( f \) satisfies the regularity conditions:

\[
f \in C^1(\mathbb{R}) \text{ and there exists a } \kappa_1 \in \mathbb{R} \text{ such that } f' \geq \kappa_1.
\]  

(3)

We also impose that \( f \) satisfies the following growth condition: there exist constants

\[
p > q \geq 1, \quad \alpha \geq 0, \quad \text{and} \quad 0 < \rho \leq \beta
\]

such that

\[
\rho|s|^{p+1} \leq sf(s) \leq \alpha|s|^{q+1} + \beta|s|^{p+1} \quad \forall s \in \mathbb{R}.
\]

(4)

Furthermore, let

\[
F(s) := \int_0^s f(\tau)d\tau.
\]

Theorem 1. Assume \( f \) satisfies (3) and (4). Let \( k < 2 \),

\[
a = \rho \left( \frac{2}{\rho(p+1)} \right)^{\frac{p+1}{p-1}} - \left( \frac{2}{\rho(p+1)} \right)^{\frac{2}{p-1}}, \quad \gamma_2 = \frac{\alpha(p+1) + \beta(q+1)}{(q+1)(p+1)\rho},
\]

\[
\mu_3 \in \begin{cases} (0, \frac{2-k}{k}), & k \in (0, 2) \\ (0, \infty), & k \leq 0 \end{cases}, \quad c \in \begin{cases} (0, \frac{2-1+k}{\gamma_2}), & k \in (0, 2) \\ (0, \infty), & k \leq 0 \end{cases}
\]

If \( w(t) \) is a local solution to (2) that satisfies

\[
\frac{k}{2} w'(0)^2 + w'(0)w'''(0) + F(w(0)) - \frac{1}{2} w''(0)^2 > \frac{\alpha}{q+1} - \frac{a}{c},
\]

(5)

then \( w \) blows up in finite time for \( t > 0 \).

Remark 1 (Conditions on the initial data). The condition (5) is in the spirit of [10, Eq. (12)]; however, here we define it in terms of the energy \( E \) (introduced below in (10)), which is constant in time and therefore is a more natural invariant for the problem, obtained in fact through the classical energy multiplier \( w' \) applied to (2). Instead [10, see (12) and p. 20] employs a different nonconstant convex invariant to characterize the initial data. Global existence for other data sets is under investigation; numerical evidence suggests that nontrivial solutions may exist globally.
in time. For example, see Fig. 1 where the solution does not satisfy (5) and appears to be stable and doubly-periodic.

Using linear transformations we are able to prove blow-up results for \( k \in \mathbb{R} \) provided the nonlinearity \( f \) satisfies certain conditions.

**Corollary 1.** Let \( c_1, c_2 \neq 0 \) and \( c_3, r \in \mathbb{R} \) be such that \( r/c_2^2 < 2 \). Let \( g \) be chosen so that

\[
 f(s) := \frac{1}{c_1c_2^2} g\left(c_1[s + c_3]\right) \tag{6}
\]

where \( f \) satisfies conditions (3) and (4). Let \( a, \gamma_2, \mu_2, \) and \( c \) be defined as in Theorem 1. Suppose \( u \) solves

\[
 u'''' + ru'' + g(u) = 0 \tag{7}
\]

with initial conditions satisfying

\[
 \frac{r}{2c_1^2c_2^4}u''(0)^2 + \frac{1}{c_1^2c_2^4}u''(0)u'''(0) + F\left(\frac{1}{c_1}u(0) - c_3\right) - \frac{1}{2c_1^2c_2^4}u''(0)^2
\]

\[
 > \frac{\alpha}{q + 1} - \frac{\alpha}{c}. \tag{8}
\]

Then \( u \) blows up in finite time on the right-maximal existence interval \([0, \omega)\) if \( c_2 > 0 \), and on the left-maximal interval \((-\omega, 0]\) if \( c_2 < 0 \).

(See Example 1 below for an application.)

**Proof.** Let

\[
 w(t) := \frac{1}{c_1}u\left(\frac{t}{c_2}\right) - c_3
\]

so that \( u(t) = c_1(w(c_2t) + c_3) \). Rewrite Eq. (7) in terms of \( w \):
\[ 0 = u'''(t) + ru''(t) + g(w(t)) = c_1 c_2^4 w'''(c_2 t) + r c_1 c_2^2 w''(c_2 t) + g\left(c_1 w(c_2 t) + c_3\right) = c_1 c_2^4 w'''(c_2 t) + r c_1 c_2^2 w''(c_2 t) + c_1 c_2^4 f\left(w(c_2 t)\right) \text{ using (6)}. \]

Dividing by \(c_1 c_2^4\) results in
\[ 0 = w'''(c_2 t) + r c_1 c_2^2 w''(c_2 t) + f\left(w(c_2 t)\right), \quad t \geq 0. \]

Set \(k = r/c_2^2 < 2.\) If \(u\) satisfies (7) on some neighborhood \((-C, C)\) of 0, then for \(\tau = c_2 t,\) the function \(w\) satisfies
\[ w'''(\tau) + k w''(\tau) + f\left(w(\tau)\right) = 0 \]
on the interval \((-C/|c_2|, C/|c_2|).\) The restriction on the initial conditions of \(u\) from Eq. (8) implies
\[ \frac{k}{2} w'(0)^2 + w'(0) w'''(0) + F\left(w(0)\right) - \frac{1}{2} w''(0)^2 > \frac{\alpha}{q + 1} - \frac{a}{c}. \]
The conclusion now follows from Theorem 1. \(\square\)

**Example 1.** Consider a function \(w(t)\) satisfying
\[ u'''(t) + ru''(t) + |u(t)|^b u(t) = 0 \]
with \(r > 0\) and \(b > 0.\) If we pick \(c_1 = 1, c_2 = \sqrt{r}, c_3 = 0,\) notice from (6) we have that
\[ f(s) := \frac{1}{r^2} g(s) = \frac{1}{r^2} |s|^b s \]
satisfies conditions (3) and (4) and \(r/c_2^2 = 1 < 2.\) Corollary 1 now gives a set of initial conditions where \(u\) will blow up. In particular, when
\[ \frac{1}{2r} u'(0)^2 + \frac{1}{r^2} u'(0) u'''(0) + \frac{|u(0)|^{b+2}}{r^2(b + 2)} - \frac{1}{2r^2} u''(0)^2 > \frac{\alpha}{q + 1} - \frac{a}{c}. \]
Corollary 1 can also be used to tackle some nonlinearities not satisfying conditions (3) and (4). The next example will look at the nonlinearity \(f(w) = w^3 - 3w^2 + 4w - 2\) which certainly doesn’t satisfy (4).

**Example 2.** Consider a function \(u(t)\) satisfying
\[ u'''(t) + ru''(t) + u^3 - 3u^2 + 4u - 2 = 0 \]
where \(r > 0.\) Set \(c_1 = 1, c_2 = \sqrt{r}, c_3 = 1.\) Then
satisfies conditions (3) and (4). Corollary 1 now prescribes initial conditions for which \( u \) will blow up in finite time. In particular, \( u \) blows up when

\[
\frac{1}{2r} u'(0)^2 + \frac{1}{r^2} u'(0) u''(0) + \frac{1}{r^2} \left( \frac{(u(0) - 1)^3}{3} + \frac{(u(0) - 1)^2}{2} \right) = \frac{1}{2r^2} u''(0)^2 > \frac{\alpha}{q + 1} - \frac{a}{c}.
\]

2.1. Outline of the proof to Theorem 1

The proof of the main Theorem 1 relies on three main components:

- The first step, to show that global solutions to (2) cannot be eventually of one sign, has been settled for \( k \geq 0 \) in [3] under suitable conditions on \( f \). In [10] an oscillation result was proven for \( k \leq 0 \). The only contribution to this step by the present paper is a modified proof of the oscillation result for \( k \leq 0 \) which holds under relaxed differentiability assumptions on \( f \).
- Inspired by the approach in [10], we introduce certain energy functions in Sections 3, which are studied in Section 4. We use the energy functions to prove in Section 5 that extremum values for the solution grow at least geometrically fast.
- Using the results of Section 5 we show in Section 6 that distances between consecutive zeros of a solution \( w \) form a summable sequence, therefore \( |w| \) necessarily blows up in finite time. To accomplish this we employ the following properties, derived in Section 5, of a solution of (2) satisfying the hypothesis of Theorem 1:
  - (W1) \( w \) changes sign at zeros and is never eventually of one sign.
  - (W2) On an interval of one sign \([z_i, z_{i+1}]\) where \( w(z_i) = w(z_{i+1}) = 0 \), there is exactly one extremum \( m_i \).
  - (W3) On \([z_i, m_i]\) and \([m_i, z_{i+1}]\), \( |w| \) is nondecreasing and nonincreasing respectively.
  - (W4) On \([m_i, z_{i+1}]\) we know \( w \) is concave down on an interval of positivity and concave up on an interval of negativity.

For an example of a numerically obtained solution see Fig. 2. Notice how the peaks increase in magnitude while the zeros converge to a limit point.

The plot in Fig. 3 exhibits the behavior between consecutive zeros on a positivity interval and the geometric features (W1)–(W4) of the solution.

3. Summary of constants and energy functions

In this section we summarize for easy reference all parameters and energy functions that will be used to prove the main theorem. Recall from condition (4) that \( \rho, p, \) and \( q \) were constants used to quantify the growth of \( f \). If we let \( a \) be the minimum of the polynomial \( \rho |x|^{p+1} - x^2 \) then

\[
a := \rho \left( \frac{2}{\rho(p + 1)} \right)^{p+1} - \left( \frac{2}{\rho(p + 1)} \right)^2.
\]
Fig. 2. Blow-up through oscillations. This numerically obtained function was rescaled vertically to exhibit more peaks.

Fig. 3. Numerically obtained solution of (2) for: \( k = 3.5, \ f(t) = t^5, \) and \([w(0), w'(0), w''(0), w'''(0)] = [0, 0.5, 0.3, -0.6].\) \( z_i, z_{i+1}: \) consecutive zeros of the solution. \( m_i: \) unique extremum on the interval of positivity. \( r_i: \) first inflection point after the zero \( z_i. \) ---: \( w(t), \ -\ -\ -: \ w'(t), \ \cdots\cdots: \ w''(t).\)

Parameters \( \gamma_1, \gamma_2, \xi_1, \xi_2 \) will describe the growth of \( F(s) := \int_{0}^{s} f(t) d\tau \) in (14), Lemma 4, and they are defined by

\[
\gamma_1 := \frac{\rho}{(p + 1)(\alpha + \beta)} \quad \xi_1 := -\gamma_1 \alpha
\]
\[
\gamma_2 := \frac{\alpha}{(q + 1)\rho} + \frac{\beta}{(p + 1)\rho} \quad \xi_2 := -\frac{\alpha}{q + 1}.
\]
The set of constants to follow will be used to define suitable energy functions associated with solution \( w \). We begin with \( \mu_3 \) which will be a free parameter in the interval

\[
\mu_3 \in \begin{cases} 
(0, \frac{2-k}{k}) & k \in (0, 2) \\
(0, \infty) & k \leq 0.
\end{cases}
\]

The constant \( c \) will be used to determine admissible initial energies for blow-up results. It can be set to any value in

\[
c \in \begin{cases} 
(0, \frac{2-(1+\mu_3)k}{2+k\gamma_2}) & k \in (0, 2) \\
(0, \infty) & k \leq 0.
\end{cases}
\]

In terms of the above parameters we define:

\[
\mu_1 := 2 - k \left( \frac{c}{2} + c\gamma_2 + 1 + \mu_3 \right) \quad \alpha_1 := \frac{1}{2} \left( 2 - k(\mu_3 + c\gamma_2 + 1) \right)
\]

\[
\mu_2 := \mu_3 + c \left( \frac{\gamma_2 + 3}{2} \right) \quad \alpha_2 := \frac{c}{2} + \mu_3 + c\gamma_2 + 1
\]

\[
\alpha_3 := \mu_3 + c\gamma_2 + 1.
\]  \tag{9}

It will be shown that these constants are positive. Recall that \( \kappa_1 \) was a lower bound on \( f' \) (from (3)). Using it we define

\[
\kappa_2 := \min \left\{ \frac{|\mu_2|}{|k| + 1}, \frac{|\mu_1|}{|\kappa_1| + 1} \right\}
\]

We now introduce the energy functions (and compute some of their derivatives) that will aid in exhibiting the blow-up mechanism of solutions to (2). These functionals inspired by similar constructs in [10] are able to detect the blow-up of \( w \) through their properties (e.g. convexity for \( G \) and \( H \)):

\[
E(t) := \frac{k}{2} w'(t)^2 + w(t)w'''(t) + F(w(t)) - \frac{1}{2} w''(t)^2
\]

\[
A(t) := w(t) f(w(t)) + w''(t)^2 + 2w(t)w''(t).
\]  \tag{10}

It will be shown that the above energy \( E \) remains invariant during the life-time of solutions; \( E \) and \( A \) together will be used to establish the growth and convexity properties of the following functional \( G \), which will be one of the primary tools for showing the blow-up of \( w \):

\[
G(t) := \alpha_1 w(t)^2 + \alpha_2 w'(t)^2 - \alpha_3 w(t)w''(t)
\]

\[
G'(t) = 2\alpha_1 w(t)w'(t) + (2\alpha_2 - \alpha_3)w'(t)w''(t) - \alpha_3 w(t)w'''(t)
\]

\[
G''(t) = 2\alpha_1 \left( w(t)w''(t) + w'(t)^2 \right) + 2(\alpha_2 - \alpha_3) \left( w'(t)w'''(t) + w''(t)^2 \right)
\]

\[
+ \alpha_3 \left( w''(t)^2 - w(t)w'''(t) \right).
\]  \tag{11}
In addition the following two functionals will be employed to prove the geometric growth of the extremum values of \( w \) (\( \Phi \) was previously introduced in [10]):

\[
\Phi(t) := \frac{1}{2} w''(t)^2 + F(w(t))
\]

\[
H(t) := G(t) + \kappa_2 \Phi(t)
\]

\[
= \alpha_1 w(t)^2 + \alpha_2 w'(t)^2 - \alpha_3 w(t) w''(t) + \frac{\kappa_2}{2} w''(t)^2 + \kappa_2 F(w(t)).
\]

(12)

4. Convexity of \( G \) and \( H \)

This section proves several properties of the energy functions introduced in (10)–(12), in particular the strict convexity of \( G \) and \( H \). We begin by showing that \( E \) is conserved in time.

**Lemma 2.** If \( w \) is a solution to Eq. (2) then \( E(t) = E(0) \) for all \( t \) in the interval of existence.

**Proof.** Assume \( w \) is a solution of Eq. (2), then (suppressing “\((t)\)"

\[
\frac{dE}{dt} = kw'w'' + w'w''' + w''w'' + f(w)w' - w''w''
\]

\[
= w(kw'' + w''' + f(w)) = 0. \quad \Box
\]

The next lemma introduces and derives a lower bound for the function \( A \). \( A \) will be used later to provide a lower bound on \( G'' \).

**Lemma 3.** Assume \( f \) satisfies condition (4) and

\[
A(t) := w(t) f\left(w(t)\right) + w''(t)^2 + 2w(t)w''(t).
\]

Then

\[
A(t) \geq a := \rho \left( \frac{2}{\rho(p+1)} \right)^{\frac{p+1}{p-1}} - \left( \frac{2}{\rho(p+1)} \right)^{\frac{2}{p-1}}
\]

for all \( t \) in the interval of existence.

**Proof.** We estimate directly:

\[
A = wf(w) + \left(w''\right)^2 + 2w w'' \geq \rho w^{p+1} + \left(w''\right)^2 + 2w w'' \quad \text{ (condition (4))}
\]

\[
= \rho w^{p+1} - w^2 + \left(w''\right)^2 + 2w w''
\]

\[
\geq \rho w^{p+1} - w^2 \quad \text{ (Young’s inequality)}
\]

\[
\geq a.
\]

The last line follows by minimizing \( \rho |x|^{p+1} - x^2 \) with \( x \in \mathbb{R} \). \( \Box \)
The next result describes the growth of $F$ and will be used later to show the convexity of the energy $G$.

**Lemma 4** *(Growth of $F$).* Assume $f$ satisfies condition (4) (with parameters $\alpha, \beta, p, q$). Let $F(s) := \int_0^s f(\tau)d\tau$, and

\[
\gamma_1 = \frac{\rho}{(p + 1)(\alpha + \beta)}, \quad \zeta_1 = -\gamma_1 \alpha, \quad \gamma_2 = \frac{(p + 1) + \beta(q + 1)}{(q + 1)(p + 1)\rho}, \quad \zeta_2 = \frac{\alpha}{q + 1},
\]

then for all $s \in \mathbb{R}$

\[
\frac{\rho}{p + 1}|s|^{p + 1} \leq F(s) \leq \frac{\alpha}{q + 1}|s|^{q + 1} + \frac{\beta}{p + 1}|s|^{p + 1}.
\]

(13)

Consequently, by (4) for all $t$ in the interval of existence

\[
\gamma_1 w(t)f(w(t)) + \zeta_1 \leq F(w(t)) \leq \gamma_2 w(t)f(w(t)) + \zeta_2.
\]

(14)

**Proof.** Recall that $p > q$. Then for all $s$ (note that both $s \geq 0$ and $s < 0$ yield the same result via lower bound in (4)) we have

\[
F(s) = \int_0^s f(\tau)d\tau \geq \int_0^s \rho \tau |\tau|^{p - 1}d\tau
\]

\[
= \frac{\rho}{p + 1}|s|^{p + 1} = \gamma_1(\alpha + \beta)|s|^{p + 1}
\]

\[
\geq \gamma_1 \alpha(1 + |s|^{p + 1}) + \gamma_1 \beta |s|^{p + 1} - \gamma_1 \alpha
\]

\[
\geq \gamma_1 \alpha |s|^{q + 1} + \gamma_1 \beta |s|^{p + 1} + \zeta_1
\]

\[
\geq \gamma_1 s f(s) + \zeta_1.
\]

We find the upper bound in a similar manner (again both $s \leq 0$ and $s > 0$ yield the same inequality using the upper bound in (4)):

\[
F(s) = \int_0^s f(\tau)d\tau \leq \int_0^s \alpha \tau |\tau|^{q - 1} + \beta \tau |\tau|^{p - 1}d\tau
\]

\[
= \frac{\alpha}{q + 1}|s|^{q + 1} + \frac{\beta}{p + 1}|s|^{p + 1}
\]

\[
\leq \frac{\alpha}{q + 1}(1 + |s|^{p + 1}) + \frac{\beta}{p + 1}|s|^{p + 1}
\]

\[
= \frac{\alpha}{q + 1} + \left(\frac{\alpha}{q + 1} + \frac{\beta}{p + 1}\right)|s|^{p + 1}
\]

\[
= \zeta_2 + \gamma_2 \rho |s|^{p + 1}
\]

\[
\leq \zeta_2 + \gamma_2 s f(s).
\]
Recall from Section 3 the definition of the energy function $G$:

$$G(t) = \alpha_1 w(t)^2 + \alpha_2 w'(t)^2 - \alpha_3 w(t)w''(t)$$

where coefficients $\alpha_i$ satisfy

$$\alpha_1 = \frac{1}{2} \left( 2 - k \left( \mu_3 + c \gamma_2 + 1 \right) \right), \quad \alpha_2 = \frac{c}{2} + \mu_3 + c \gamma_2 + 1.$$ 

The next lemma will show that by placing certain restrictions on the initial energy we can ensure that $G''$ is bounded below by a positive constant.

**Lemma 5 (Strict convexity of $G$).** Assume $w$ is a local solution to Eq. (2) and $f$ satisfies condition (4). Let $\varepsilon > 0$, $k < 2$, 

$$\mu_3 \in \begin{cases} (0, \frac{2-k}{k}), & k \in (0, 2) \\ (0, \infty), & k \leq 0, \end{cases} \quad c \in \begin{cases} (0, \frac{2-(1+\mu_3)k}{\frac{c}{2}+k\gamma_2}), & k \in (0, 2) \\ (0, \infty), & k \leq 0, \end{cases}$$

$$\mu_1 = 2 - k \left( \frac{c}{2} + c \gamma_2 + 1 + \mu_3 \right), \quad \mu_2 = \mu_3 + c \gamma_2 + \frac{3c}{2}.$$ 

Then $\mu_1, \mu_2, \mu_3, \alpha_1, \alpha_2, \alpha_3 > 0$ and if $E(0)$ satisfies $cE(0) > \xi_2 c - a$ then for some $\varepsilon > 0$ we have

$$G'' \geq \varepsilon + \mu_1 (w')^2 + \mu_2 (w'')^2 + \mu_3wf(w)$$

on the interval of existence of the solution.

**Proof.** First let us verify the positivity of the constants. If $k \leq 0$ then all above constants are trivially positive; therefore, let us consider $k \in (0, 2)$. It is clear that $\mu_3$ and $c$ exist and are positive. Because $\gamma_2 > 0$ we have $\mu_2, \alpha_2, \alpha_3 > \mu_3 > 0$. Next, we focus on $\mu_1$. Recall $c$ satisfies

$$0 < c < \frac{2 - (1 + \mu_3)k}{\frac{c}{2} + k\gamma_2}$$

so that

$$\frac{ck}{2} + ck\gamma_2 < 2 - k - \mu_3k$$

and

$$0 < 2 - \frac{ck}{2} - ck\gamma_2 - k - \mu_3k = \mu_1.$$ 

Since $2\alpha_1 = 2 - k\alpha_3 = \mu_1 + \frac{ck}{2}$ we have that $\alpha_1 > 0$ as well.
Now we will establish a lower bound on $G''$. Recall from (11) that

$$ G'' = 2\alpha_1 (ww'' + (w')^2) + 2(\alpha_2 - \alpha_3)(w'w''' + (w'')^2) + \alpha_3 ((w'')^2 - ww'''). $$

Since $cE(0) > \xi_2 c - a$ pick $\varepsilon > 0$ such that $cE(0) \geq \xi_2 c - a + \varepsilon$. Utilizing the lower bound on the energy function $A(t)$ from Lemma 3, we obtain

$$ c\xi_2 + \varepsilon \leq cE(0) + a 
\leq cE(t) + A(t) 
= \frac{ck}{2} (w')^2 + cw'w''' + \left[ cF(w) \right] - \frac{c}{2} (w'')^2 + w f(w) + (w'')^2 + 2ww'' 
\leq \frac{ck}{2} (w')^2 + cw'w''' + \left[ c\gamma_2 w f(w) + ck\xi_2 \right] - \frac{c}{2} (w'')^2 
+ wf(w) + (w'')^2 + 2ww''. $$

Subtracting $c\xi_2$ from each side, and adding and subtracting $\mu_3 w f(w)$ results in

$$ \varepsilon \leq \frac{ck}{2} (w')^2 + cw'w''' + c\gamma_2 w f(w) - \frac{c}{2} (w'')^2 + w f(w) + (w'')^2 + 2ww'' 
+ \mu_3 w f(w) - \mu_3 w f(w) 
= \frac{ck}{2} (w')^2 + cw'w''' - ck\gamma_2 w w'' - c\gamma_2 w w''' - \frac{c}{2} (w'')^2 - kw w'' - w w''' 
\left[ \begin{array}{c} w f(w) = c\gamma_2 w (-k w'' - w''' ) \\
\mu_3 w f(w) = \mu_3 w (-k w'' - w''' ) 
\end{array} \right] 
+ (w'')^2 + 2ww'' - \mu_3 kww'' - \mu_3 w w''' - \mu_3 w f(w). $$

Combining like terms and using arithmetic manipulations yields

$$ \varepsilon \leq \frac{ck}{2} (w')^2 + cw'w''' + (-ck\gamma_2 - k + 2 - \mu_3 k) w w'' - \left( c\gamma_2 + 1 + \mu_3 \right) w w''' 
+ \left( 1 - \frac{c}{2} \right) (w'')^2 - \mu_3 w f(w) 
= \frac{ck}{2} (w')^2 + cw'w''' + (-ck\gamma_2 - k + 2 - \mu_3 k) w w'' - \alpha_3 w w''' + \left( 1 - \frac{c}{2} \right) (w'')^2 
- \mu_3 w f(w) + c \left[ (w'')^2 \right]_{0} + c \left[ (w'')^2 \right]_{0} + c \left( (w'')^2 \right)_{0} 
+ \alpha_3 \left[ (w'')^2 \right]_{0} 
+ \alpha_3 \left[ (w'')^2 \right]_{0} \left[ (w''')^2 \right]_{0} $$
\[
\begin{align*}
&= c(w' w'' + (w'')^2) + (-ck\gamma_2 - k + 2 - \mu_3)(ww'' + (w')^2) + \alpha_3((w'')^2 - w w''') \\
&\quad + \left(\frac{ck}{2} + ck\gamma_2 + k - 2 + \mu_3 k\right)(w)^2 + \left(1 - c - c\gamma_2 - \frac{c}{2} - 1 - \mu_3\right)(w'')^2 \\
&\quad - \mu_3 w f(w) \\
&= 2(\alpha_2 - \alpha_3)(w' w'' + (w'')^2) + 2\alpha_1(ww'' + (w')^2) + \alpha_3((w'')^2 - w w''') \\
&\quad - \mu_1(w')^2 - \mu_2(w'')^2 - \mu_3 w f(w) \\
&= G'' - \mu_1(w')^2 - \mu_2(w'')^2 - \mu_3 w f(w).
\end{align*}
\]

We conclude

\[
G'' \geq \varepsilon + \mu_1(w')^2 + \mu_2(w'')^2 + \mu_3 w f(w) \geq \varepsilon. \quad \square
\]

The proof of Theorem 1 will rely on the fact that $G$ is convex and therefore we will frequently appeal to the following condition on the initial energy $E(0)$ (note also that since $c, \zeta_2 \geq 0$ and $a \leq 0$ we infer $E(0) > 0$):

\[
c E(0) > \zeta_2 c - a > 0 \quad (16)
\]

which is precisely the condition (5) in the hypothesis of Theorem 1.

The next result will show that $H$ is convex.

**Lemma 6 (Strict convexity of $H$).** Let $w$ be a nontrivial local solution to Eq. (2) with $k < 2$. Assume $f$ satisfies conditions (3) and $E(0)$ satisfies (16). Let

\[
\Phi(t) := \frac{1}{2} w''(t)^2 + F(w(t)) \quad \text{and} \quad H(t) := G(t) + \kappa_2 \Phi(t)
\]

where $\kappa_2 > 0$ is given by

\[
\kappa_2 := \min\left\{\frac{|\mu_2|}{|k| + 1}, \frac{|\mu_1|}{|\kappa_1| + 1}\right\} \quad (17)
\]

Then $H''(t) \geq \varepsilon$, for some $\varepsilon > 0$.

**Proof.** By condition (3) we know $f' > \kappa_1$. We differentiate $\Phi$ twice to find

\[
\Phi''(t) = (w'')^2 - k(w'')^2 + f'(w)(w')^2 \geq -k(w'')^2 + \kappa_1(w')^2.
\]

Recall from Lemma 5 that

\[
G'' \geq \mu_1(w')^2 + \mu_2(w'')^2 + \varepsilon
\]

since $\mu_3 w f(w) \geq 0$. Thus,
\[ H'' = \kappa_2 \Phi'' + G'' \]
\[ \geq -k \kappa_2 (w'')^2 + \kappa_2 \kappa_1 (w')^2 + \mu_1 (w')^2 + \mu_2 (w'')^2 + \varepsilon \]
\[ \geq -\left| \frac{\mu_2}{|k| + 1} \right| (w'')^2 - \left| \frac{\mu_1}{|k_1| + 1} \right| (w')^2 + \mu_1 (w')^2 + \mu_2 (w'')^2 + \varepsilon \]
\[ \geq \varepsilon. \quad \square \]

Now that we have convexity results for \( G \) and \( H \) we can infer several interesting properties of \( w \) that will be discussed in the next section.

5. Growth of \( G \) and behavior of \( w \)

The convexity results in Section 4 prove that both \( G \) and \( H \) are eventually strictly increasing. This fact leads to several interesting properties of hypothetical global solutions to (2). As Section 6 will subsequently demonstrate, these conditions lead to a contradiction implying that global solutions to (2) cannot exist. The first result we prove is that \( G(t) \) grows exponentially for \( t \) sufficiently large.

**Lemma 7.** Assume \( f \) satisfies (4), \( k < 2 \), \( w \) is a global solution to (2), and \( E(0) \) satisfies (16). For some \( T \geq 0 \), there exist constants \( \theta > 0 \), \( C_1 > 0 \), and \( C_2 \in \mathbb{R} \) such that

\[ G(t) \geq C_1 e^{\theta t} + C_2 \quad \text{for all } t \geq T. \]

**Proof.** Since \( p > 1 \), \( \mu_3 \), \( \varepsilon > 0 \), there exists \( C_3 > 0 \) such that

\[ C_3 (\mu_3 \rho |w|^{p+1} + \varepsilon) \geq \left( \alpha_1 + \frac{\alpha_3}{2} \right) w^2. \]

Recall \( \mu_1, \mu_2, \mu_3 > 0 \). Find \( C_4 > 0 \) such that

\[ C_4 \geq \max \left\{ C_3, \frac{\alpha_2}{\mu_1}, \frac{\alpha_3}{2 \mu_2} \right\}. \]

Then

\[ C_4 G'' \geq C_4 \varepsilon + C_4 \mu_3 w f (w) + C_4 \mu_1 (w')^2 + C_4 \mu_2 (w'')^2 \]
\[ \geq C_4 (\varepsilon + \mu_3 \rho |w|^{p+1}) + C_4 \mu_1 (w')^2 + C_4 \mu_2 (w'')^2 \]
\[ \geq \left( \alpha_1 + \frac{\alpha_3}{2} \right) w^2 + \alpha_2 (w')^2 + \frac{\alpha_3}{2} (w'')^2 \]
\[ = \alpha_1 w^2 + \alpha_2 (w')^2 + \alpha_3 \left( \frac{w^2}{2} + \frac{(w'')^2}{2} \right) \]
\[ \geq \alpha_1 w^2 + \alpha_2 (w')^2 - \alpha_3 w w'' \]
\[ = G. \]
Because $G''$ is strictly positive by Lemma 4, we can find a time $T > 0$ such that $G(t) \geq 0$ and $G'(t) \geq 1$ for $t \geq T$. Then for $t \geq T$ we know $G$ is bounded below by the solution to the initial value problem:

$$C_4 u''(t) = u(t) \quad \text{for } t \geq T, \quad \text{with } u(T) = 0, u'(t) = 1.$$ 

Hence for $t \geq T$,

$$G(t) \geq C_1 e^{\theta t} + C_2. \quad \Box$$

The next result provides a lower exponential bound for the growth of $w(t)$ at extrema when $t$ is taken sufficiently large.

**Lemma 8 (Growth of $w$ at the extrema).** Assume $w$ is a nontrivial global solution to Eq. (2), $f$ satisfies condition (4), and $E(0)$ satisfies (16). There exist a $T \geq 0$ and positive constants $C$, $r$ such that for any local extremum $m \geq T$ of $w$ we have

$$|w(m)| \geq C e^{r m}.$$

**Proof.** Since $w'(m) = 0$ we have

$$E(m) = F(w(m)) - \frac{1}{2} w''(m)^2 = E(0)$$

$$G(m) = \alpha_1 w(m)^2 - \alpha_3 w(m) w''(m).$$

Solving for $|w''(m)|$ in the energy equation yields

$$|w''(m)| = \sqrt{2 F(w(m)) - 2E(0)}.$$ 

Notice that at a local extremum $m$ we always have $w(m) w''(m) \leq 0$ and consequently,

$$G(m) = \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{2 F(w(m)) - 2E(0)}.$$ 

Thus by bound (13) on $F$ from Lemma 4, and using $p > q$

$$G(m) \leq \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{\frac{2\alpha}{q+1} |w(m)|^{q+1} + \frac{2\beta}{p+1} |w(m)|^{p+1}}$$

$$\leq c_1 + c_2 |w(m)|^{\frac{p+3}{2}}$$

where $c_1, c_2 > 0$ are sufficiently large. By Lemma 7 we know there is a $T$ such that for $m \geq T$ we have $G(m) \geq C_1 e^{\beta m} + C_2$ for some constants $C_1, C_2 > 0$. Consequently,

$$|w(m)| \geq \left( \frac{C_1 e^{\beta m} + C_2 - c_1}{c_2} \right)^{\frac{2}{p+3}}$$
and for sufficiently large \( T \) (and consequently large \( m \)),

\[
|w(m)| \geq \left( \frac{C_1}{2c_2} \right)^{\frac{2}{p+3}} e^{\frac{2p}{p+3} m}. \quad \Box
\]  

(19)

We will discover that \( w \) is never eventually of one sign. This coupled with the above lemma allows us to conclude that \( w \) is unbounded, at least asymptotically. Next we will look at the value of \( G(t) \) at an extremum, \( m \), of \( w \). The next lemma will show that for \( m \) sufficiently large, \( G(m) \) is comparable to \( |w(m)|^{\frac{p+3}{2}} \).

**Lemma 9 (Bounds on \( G(m) \)).** Assume \( f \) satisfies condition (4). If \( w \) is a nontrivial global solution to (2) with initial conditions satisfying (16), then there exist nonnegative constants \( T, C_1, \) and \( C_2 \), such that if \( m \) is a local extremum of \( w \) and \( m \geq T \), then

\[
C_1 |w(m)|^{\frac{p+3}{2}} \leq G(m) \leq C_2 |w(m)|^{\frac{p+3}{2}}.
\]

**Proof.** The upper bound follows as in the proof of Lemma 8, from (18) and the fact that \( w(m) \) is eventually large, as dictated by (19).

For the lower bound, again, since \( m \) is an extremum, we know \( w(m)w''(m) \leq 0 \) and

\[
E(m) = F(w(m)) - \frac{1}{2} w''(m)^2 \quad \iff \quad |w''(m)| = \sqrt{2F(w(m)) - 2E(0)}. \quad (20)
\]

Thus for a sufficiently large \( m \),

\[
G(m) = \alpha_1 w(m)^2 - \alpha_3 w(m)w''(m)
\]

\[
= \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{2F(w(m)) - 2E(0)} \quad \text{(Eq. (20))}
\]

\[
\geq \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{\frac{2\rho}{p+1} |w(m)|^{p+1} - 2E(0)} \quad \text{(Lemma 4)}
\]

\[
\geq \alpha_1 w(m)^2 + \alpha_3 |w(m)| \sqrt{\frac{\rho}{p+1} |w(m)|^{p+1}} \quad \text{(use Lemma 8 and large \( m \))}
\]

\[
\geq \alpha_3 \left( \frac{\rho}{p+1} \right)^{\frac{1}{2}} |w(m)|^{\frac{p+3}{2}} \quad \text{(recall \( p > 1 \))}
\]

\[
= C_2 |w(m)|^{\frac{p+3}{2}}. \quad \Box
\]

5.1. Oscillatory behavior of \( w \)

Following the approach of [10], the proof of the main result is based on the fact that \( w \) cannot remain of one sign and exist globally.

The following result comes from [3, Thm. 4]. For our purposes, it states that if \( k \geq 0 \) and \( f \) satisfies conditions (3) and (4), then nontrivial global solutions to (2) are never eventually of one sign.
**Theorem 10** (Oscillations of \(w\) when \(k \geq 0\)). (See [3, Thm. 4].) Let \(k \geq 0\) and suppose

\[ f \in \text{Lip}_{\text{loc}}(\mathbb{R}), \quad f(t) > 0 \text{ for every } t \in \mathbb{R} \setminus \{0\}. \]

If \(w\) is a nontrivial global solution to (2), then \(w(t)\) changes sign infinitely many times as \(t \to \infty\) and as \(t \to -\infty\).

First we need the following lemma that will also come in handy later on when proving some geometric properties of the graph of \(w\).

**Lemma 11** (Decreasing \(|w|\) after an extremum). Assume \(f\) satisfies (3) and (4), \(k < 2\), and \(w\) is a nontrivial global solution to (2) with initial conditions satisfying (16). Let \((z_i, z_{i+1})\) be an interval where \(w\) is of one sign. There exists a \(T \geq 0\) such that if \(z_i \geq T\), \(m_i \in (z_i, z_{i+1})\) is a zero of \(w'\), and \(|w|\) is strictly decreasing on some interval \((m_i, m_i + \delta) \subset (m_i, z_{i+1})\) with \(\delta > 0\), then \(w' \neq 0\) on \((m_i, z_{i+1})\).

**Proof.** To the contrary, assume \(n_i \geq m_i + \delta\) is the next point where \(w'(n_i) = 0\) in \((m_i, z_{i+1})\). By assumption \(|w(n_i)| < |w(m_i)|\). If in addition \(w(n_i) = 0\), then the energy satisfies \(E(n_i) = -\frac{1}{2} w''(n_i)^2 \leq 0\) contradicting (16) which requires \(E(0) > 0\). Thus \(w(n_i) \neq 0\), in particular \(n_i < z_{i+1}\). Since \(|w|\) is decreasing till \(n_i\), we know \(n_i\) cannot be a point of maximum of \(|w|\) and consequently

\[ w(n_i)w''(n_i) \geq 0. \]

Now recall,

\[ G(n_i) = \alpha_1 w(n_i)^2 - \alpha_3 w(n_i)w''(n_i) \]
\[ G(m_i) = \alpha_1 w(m_i)^2 - \alpha_3 w(m_i)w''(m_i). \]

Provided \(m_i\) is sufficiently large, **Lemma 5** implies \(G(n_i) > G(m_i)\); hence,

\[ \alpha_1 w(n_i)^2 - \alpha_3 w(n_i)w''(n_i) > \alpha_1 w^2(m_i) - \alpha_3 w(m_i)w''(m_i). \]

This implies that

\[ 0 > \alpha_1 w^2(n_i) - \alpha_1 w^2(m_i) > \alpha_3 w(n_i)w''(n_i) - \alpha_3 w(m_i)w''(m_i). \]

It follows that \(w(m_i)w''(m_i) > w(n_i)w''(n_i)\). However, \(w(m_i)w''(m_i) \leq 0\) and so \(w(n_i)w''(n_i) < 0\) contradicting \(w(n_i)w''(n_i) \geq 0\). We conclude \(w' \neq 0\) on \((m_i, z_{i+1})\). \(\square\)

The oscillation result for the case \(k \leq 0\) was proven in [10] using the assumption that \(f\) is twice continuously differentiable away from 0 and imposing some restrictions on \(f''\). Using the ideas in [3,10] we provide a modified proof (see Appendix A) of this result by requiring assumptions on only one derivative of \(f\), consequently requiring less regularity on \(w\).
Lemma 12. (Oscillations of \( w \) when \( k \leq 0 \), cf. [10].) Assume \( f \) satisfies conditions (3) and (4), \( k \leq 0 \), and \( w \) is a global solution of (2) with initial conditions satisfying (16). Then \( w \) is never eventually of one sign.

Proof. See Appendix A.

5.2. Shape of the graph and geometric growth

In the last section we learned that provided certain conditions were met, a solution \( w \) of (2) is never eventually of one sign. In this section we establish the behavior of \( w \) between consecutive zeros. We also gather previous results here to give a general picture of \( w \) in a region of one sign. In the first lemma we discover only one extremum can exist on an interval of one sign.

Lemma 13 (Single extremum on an interval of one sign). Assume \( f \) satisfies conditions (3) and (4) and \( w \) is a nontrivial global solution to Eq. (2) with initial conditions satisfying (16). There exists a \( T \geq 0 \) such that if \( w(z_i) = w(z_{i+1}) = 0 \) for \( z_i \geq T \), and \( (z_i, z_{i+1}) \) is an interval where \( w \) is of one sign, then there exists exactly one local maximum of \( |w| \) on \( [z_i, z_{i+1}] \); moreover, \( |w| \) is increasing on \( [z_i, m_i] \) and decreasing on \( [m_i, z_{i+1}] \).

Proof. Either by Lemma 11, or just observing that \( G'(z_i) = \alpha_2 w'(z_i)^2 \) and \( G' \) is eventually positive, we know that \( w'(z_i) \neq 0 \). So \( |w| \) is increasing on some interval \([z_i, z_i + \varepsilon]\). Let \( m_i \) be the first place in \([z_i, z_{i+1}]\) such that \(|w|\) is not increasing on some interval \([m_i, m_i + \varepsilon]\). If \(|w|\) is decreasing on some \([m_i, m_i + \delta], \delta \leq \varepsilon\), then we are done by Lemma 11. We know \( w \) is not constant anywhere as solutions are unique, in particular, by continuity there cannot be a dense subset of zeros of \( w' \). We also know that we cannot have a sequence of isolated points \( \{n_j\} \) converging to \( m_i \) from the right with \( w'(n_j) = 0 \) unless \(|w|\) is increasing on each interval \([n_{j+1}, n_j]\), as for \( j \) large that would contradict the assumption that \(|w|\) is nonincreasing \([m_i, m_i + \varepsilon]\). We conclude there is exactly one local maximum of \(|w|\) on \([z_i, z_{i+1}]\) and the monotone behavior before and after \( m_i \) on \([z_i, z_{i+1}]\) follows.

There are several consequences of Lemmas 11 and 13 that are of use to us. Provided \( T \) is sufficiently large and the hypotheses of the two lemmas hold, we know

- \( w \) changes sign at zeros.
- \( w \) has exactly one extremum, \( m_i \), on an interval \([z_i, z_{i+1}]\) where \( w(z_i) = w(z_{i+1}) = 0 \) and \( w \) is of one sign on \((z_i, z_{i+1})\).
- \(|w|\) is nondecreasing on \([z_i, m_i]\) and decreasing on \((m_i, z_i]\).

To simplify things, we introduce some notation. Any global nontrivial solutions to (2) satisfying the hypothesis of Theorem 10, will have infinitely many zeros, extrema, and inflection points, denoted as follows:

- \( \mathcal{Z} := \{z_i\} \) will be the zeros of \( w \) with \( z_i < z_{i+1} \).
- \( \mathcal{M} := \{m_i\} \) will be the extrema of \( w \) with \( m_i \in (z_i, z_{i+1}) \).
- \( \mathcal{R} = \{r_i\} \) where \( r_i \) is the smallest number in \([z_i, m_i]\) such that \( w''(r_i) = 0 \).
The next result will show that not only the sequence \(|w(m_i)|\) is unbounded, but eventually it grows geometrically.

**Lemma 14 (Geometric growth of \(w\) at extrema).** Assume \(f\) satisfies conditions (3) and (4) and \(w\) is a nontrivial global solution to Eq. (2) with initial conditions satisfying (16). There exists a \(T \geq 0\) such that any subsequence of \(|w(m_i)| : m_i \in \mathcal{M}, m_i \geq T\) is bounded below by the sequence \(\left\{ \frac{1}{3} \right\}^{i+1} : i \in \mathbb{N} \).

**Proof.** The argument was motivated by the approach in [10, Step 6, p. 25]. Let \(\ell_i\) be the last inflection point between \(m_{i-1}\) and \(m_i\), possibly \(m_i\) itself (one must exist because one \(m\) is a local minimum, while the other is a local maximum). Let \(T\) be sufficiently large so that the consequences of Lemma 9 are valid and Lemma 6 implies \(H'(t) > 0, H(t) > 0\).

Recalling

\[
F(w) - E = -\frac{k}{2} \left( w' \right)^2 - w''w''' + \frac{1}{2} \left( w'' \right)^2
\]

and \(w'(m_{i-1}) = 0\) results in

\[
\kappa_2 \left[ 2F(w(m_{i-1})) - E(m_{i-1}) \right] = \kappa_2 \left[ F(w(m_{i-1})) + \frac{1}{2} \left( w''(m_{i-1}) \right)^2 \right].
\]

At any extremum \(m_{i-1}\) we know \(w(m_{i-1})w''(m_{i-1}) \leq 0\) and \(w'(m_{i-1}) = 0\); therefore,

\[
\kappa_2 \left[ 2F(w(m_{i-1})) - E(m_{i-1}) \right] \leq \kappa_2 \left[ F(w(m_{i-1})) + \frac{1}{2} \left( w''(m_{i-1}) \right)^2 \right] - \alpha_3 w(m_{i-1})w''(m_{i-1}) + \alpha_1 w^2(m_{i-1})
\]

\[
= H(m_{i-1})
\]

\[
\leq H(\ell_i) \quad \text{(Lemma 6)}
\]

\[
= \alpha_1 w(\ell_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 F(w(\ell_i)). \quad (21)
\]

By Lemma 11, on \([\ell_i, m_i]\) we know \(f(w)\) and \(w'\) have the same sign, thus \(F(w(t))\) and \(w^2(t)\) are both nondecreasing; as a result from (21) we have

\[
\kappa_2 \left[ 2F(w(m_{i-1})) - E(m_{i-1}) \right] \leq \alpha_1 w(m_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 F(w(m_i)). \quad (22)
\]

By Lemma 9 (comparing growth of \(G(m_i)\) and \(w(m_i)\)),

\[
\alpha_2 w'(\ell_i)^2 \leq \alpha_1 w(\ell_i)^2 + \alpha_2 w'(\ell_i)^2 = G(\ell_i) \leq G(m_i) \leq C_1 |w(m_i)|^{\frac{p+3}{2}}. \quad (23)
\]

Since \(p > 1\) we have \(p + 1 > \frac{p+3}{2} > 2\). Furthermore \(|w(m_i)| \to \infty\) by Lemma 8. Also recall the growth estimate

\[
F(s) \geq \frac{\rho}{p+1} |s|^{p+1}
\]
from Lemma 4. These facts, Eq. (23), and $T$ taken sufficiently large show that $F(w(m_i))$ is of higher order than any of the terms appearing in (23). Hence, choosing a specific constant, \( const = \frac{\kappa_2}{2} \) (which affects the choice of $T$) we get from (23)

$$
\alpha_1 w(m_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 E(0) < \alpha_1 w(m_i)^2 + C_1 |w(m_i)|^{\frac{p+3}{2}} + \kappa_2 E(0)
$$

$$
\leq \frac{\kappa_2}{2} F(w(m_i)).
$$

Recall that $E(m_{i-1}) = E(0)$ by Lemma 3, so

$$
2\kappa_2 F(w(m_{i-1})) \leq \kappa_2 E(m_{i-1}) + \alpha_1 w(m_i)^2 + \alpha_2 w'(\ell_i)^2 + \kappa_2 F(w(m_i))
$$

$$
\leq \frac{\kappa_2}{2} F(w(m_i)) + \kappa_2 F(w(m_i)).
$$

Since $\kappa_2$, defined in (17), is positive, we conclude

$$
\frac{4}{3} F(w(m_{i-1})) \leq F(w(m_i)).
$$

Reindex $m_i$, so that $m_0 > T$. Then $\left( \frac{4}{3} \right)^F(w(m_0)) \leq F(w(m_i))$. From (13) we see that for $|s|$ large there is a constant (dependent on $\alpha, \beta, p, q$) such that

$$
F(s) \leq C|s|^{p+1}.
$$

Hence

$$
\frac{F(w(m_0))}{C} \left( \frac{4}{3} \right)^{\frac{1}{p+1}} \leq |w(m_i)|.
$$

We may assume that $F(w(m_0)) > C$ and drop the coefficient on the left, which yields the desired result. \( \square \)

The next result will show that $|w|$ is strictly concave on $[m_i, z_{i+1}]$ when $m_i$ is taken adequately large.

**Lemma 15 (Concave behavior after an extremum).** Assume $f$ satisfies conditions (3) and (4) and $w$ is a nontrivial global solution to Eq. (2) with initial conditions satisfying (16). There exists a $T \geq 0$ such that if $(z_i, z_{i+1})$ is an interval where $w$ is of one sign and $z_i \geq T$, then there are no inflection points after the extremum point $m_i$; therefore, $|w|$ is strictly concave on $[m_i, z_{i+1}]$.

**Proof.** Take $T$ to be large enough for the consequences of Lemma 11 to hold. To the contrary, assume $n \in (m_i, z_{i+1})$ is the next inflection point of $|w|$ on $(z_i, z_{i+1})$ after $m_i$. If we appeal to Lemma 5 and $T$ is sufficiently large so that $G'(t) > 0$ for all $t \geq T$ we have

$$
0 < G'(n) = 2\alpha_1 w(n)w'(n) - \alpha_3 w(n)w'''(n)
$$

$$
\Rightarrow \quad 2\alpha_1 w(n)w'(n) > \alpha_3 w(n)w'''(n).
$$
Thus $2\alpha_1 w'(n) > \alpha_3 w'''(n)$ on an interval of positivity (hence, $2\alpha_1 w'(n) < \alpha_3 w'''(n)$ on an interval of negativity). We conclude $w'''(n) < 0$ on an interval of positivity ($w'''(n) > 0$ on an interval of negativity); however, this implies, for an interval of positivity, $w'' < 0$ ($w'' > 0$ on an interval of negativity) on some interval $(n, n + \varepsilon')$ with $\varepsilon' > 0$. This contradicts though the fact that $n$ is the next inflection point after $m$. □

The next result describes the slope of $w$ at its zeros. In particular, the lemma shows that $|w'(z_i)|$ grows geometrically when $z_0$ is taken sufficiently large.

**Lemma 16 (Geometric growth of $w'$ at zeros of $w$).** Assume $w$ is a global solution to Eq. (2), $f$ satisfies conditions (3) and (4), and the initial conditions satisfy (16). There exists a $T > 0$ such that if $z_0 \geq T$, then

$$w'(z_i)^2 \geq \frac{C_1}{\alpha_2} \left( \frac{4}{3} \right)^{\frac{(p+3)}{2(p+1)}} \omega,$$

where $\{z_i\}$ is a sequence of consecutive zeros of $w$.

**Proof.** Pick $T$ sufficiently large so that $G'(t) > 0$ for $t \geq T$ (Lemma 5) and the consequences of Lemmas 9 and 14 are valid. Then

$$\alpha_2 w'(z_i)^2 = G(z_i) \geq G(m_{i-1}) \geq C_1 |w(m_{i-1})|^{\frac{p+3}{2}} \geq C_1 \left( \frac{4}{3} \right)^{\frac{(p+3)}{2(p+1)}} \omega.$$ □

Recall that $r_i \in \mathcal{R}$ denotes the first zero of $w''$ in the interval $[z_i, m_i]$. In the proof of Theorem 1 it will be necessary for $w(r_i)$ to be large. The next lemma will show that provided the inflection point is large enough, this is indeed the case.

**Lemma 17 (Geometric growth of $w$ at inflection points).** Assume $w$ is a global solution to Eq. (2), $k < 2$, $f$ satisfies conditions (3) and (4), and the initial conditions satisfy (16). There exists a $T \geq 0$ so that if $z_0 \in \mathcal{Z}$ and $z_0 \geq T$ then

$$|w(r_i)| \geq \left( \frac{\alpha_1}{\alpha_3} \right)^{\frac{1}{p+1}} C_1 \left( \frac{4}{3} \right)^{\frac{2(p+3)}{p+1}} \omega^2$$

for $r_i \in \mathcal{R}$, and for each $i \geq 0$.

**Proof.** Let $T$ be chosen so that $G'(t) > 0$ for $t \geq T$ (Lemma 5) and the conclusions of Lemmas 15 and 16 hold. Let $\delta = \frac{1}{2} \min[\mu_1, \mu_2]$. By Lemma 5,

$$\frac{d}{dt} \left( G' - \delta w^2 \right)' = G'' - 2\delta w w''$$

$$\geq \varepsilon + \mu_1 (w')^2 + \mu_2 (w'')^2 + \mu_3 f(w) - \mu_1 (w')^2 - \mu_2 (w'')^2$$
\[
= \varepsilon + \mu_3 w f(w) \geq \varepsilon.
\]
For some \( t_0 \geq T \) we have \( w'(t_0) = 0 \) by Theorem 10 and Lemma 12. Since \( G'(t) \geq 0 \) for \( t \geq t_0 \),
\[
G'(t) > \delta w'(t)^2, \quad \forall t \geq t_0. \tag{25}
\]
Henceforth without loss of generality let \( T = t_0 \). Next notice
\[
G'(r_i) = 2\alpha_1 w(r_i) w'(r_i) - \alpha_3 w(r_i) w'''(r_i).
\]
Recall \( w(r_i) \neq 0 \) as otherwise \( G'(r_i) = 0 \) contradicting Lemma 5. Solving for \( w'''(r_i) \) yields
\[
w'''(r_i) = \frac{2\alpha_1}{\alpha_3} w'(r_i) - \frac{G'(r_i)}{\alpha_3 w(r_i)}.
\]
Substitute the last identity into the definition of the energy function \( E(t) \) and recall \( E(t) = E(0) \) by Lemma 2:
\[
\frac{k}{2} w'(r_i)^2 + F(w(r_i)) + \frac{2\alpha_1}{\alpha_3} w'(r_i)^2 - \frac{w'(r_i) G'(r_i)}{\alpha_3 w(r_i)} - E(0) = 0.
\]
Hence multiplication by \( w(r_i) \) results in
\[
F(w(r_i)) w(r_i) - \frac{1}{\alpha_3} w'(r_i) G'(r_i) + \left( \frac{2\alpha_1}{\alpha_3} + \frac{k}{2} \right) w'(r_i)^2 w(r_i) - E(0) w(r_i) = 0.
\]
Assume \( r_i \) is on an interval of positivity. Consequently, we have
\[
F(w(r_i)) w(r_i) + \left[ \frac{2\alpha_1}{\alpha_3} + \frac{|k|}{2} \right] w'(r_i)^2 - E(0) w(r_i) - \frac{1}{\alpha_3} w'(r_i) G'(r_i) \geq 0. \tag{26}
\]
To simplify the subsequent analysis introduce a shorthand:
\[
x := w(r_i), \quad u := \left( \frac{2\alpha_1}{\alpha_3} + \frac{|k|}{2} \right) w'(r_i)^2 - E(0), \quad v := \frac{1}{\alpha_3} w'(r_i) G'(r_i).
\]
From Lemma 15 we have \( w'(r_i) \geq w'(z_i) \geq 0 \), then by Lemma 16 it follows that if \( r_i > z_i \geq T \) for \( T \) sufficiently, large then \( u > 0 \). Then we may rewrite Eq. (26) as
\[
x F(x) + u x - v \geq 0, \quad u > 0. \tag{27}
\]
By Eq. (25) we know that
\[
v = \frac{1}{\alpha_3} w'(r_i) G'(r_i) \geq \frac{\delta}{\alpha_3} w'(r_i)^3.
\]
Since \( u^\frac{3}{2} \) has cubic growth in \( w'(r_i) \), and the latter can be assumed sufficiently large, then there exists a constant \( y > 0 \) such that for all \( r_i \geq T \)

\[
yu^\frac{3}{2} < \frac{1}{\alpha_3} w'(r_i) G'(r_i) = v.
\]

Consequently, by (27)

\[
x F(x) + ux - yu^\frac{3}{2} > 0.
\]

Recalling the estimate on \( F \) from Lemma 4 we obtain

\[
\left( \frac{\beta}{p+1} |x|^{p+1} + \frac{\alpha}{q+1} |x|^{q+1} \right) x + ux - yu^\frac{3}{2} > 0.
\]

We claim that \( x > u^{\frac{1}{p+1}} \). To the contrary, suppose \( x \leq u^{\frac{1}{p+1}} \), then

\[
\left( \frac{\beta}{p+1} |x|^{p+1} + \frac{\alpha}{q+1} |x|^{q+1} \right) x + ux - yu^\frac{3}{2} \\
\leq \frac{\beta}{p+1} u^{\frac{p+2}{p+1}} + \frac{\alpha}{q+1} u^{\frac{q+2}{p+1}} + u^{\frac{p+2}{p+1}} - yu^\frac{3}{2}.
\]

(28)

From condition (4), we have

\[
0 < \frac{q+2}{p+1} < \frac{p+2}{p+1} < \frac{3}{2},
\]

hence the right side of (28) cannot be positive for large \( u \). This yields a contradiction and so we assume on an interval of positivity with \( r_i \geq T \) that

\[
x = w(r_i) \geq u^{\frac{1}{p+1}} = \left( \left( \frac{2\alpha_1}{\alpha_3} + \frac{|k|}{2} \right) w'(r_i)^2 - E(0) \right)^{\frac{1}{p+1}}.
\]

(29)

By Lemma 16 we know that for \( T \) large

\[
\frac{\alpha_1}{\alpha_3} w'(r_i)^2 - E(0) \geq 0.
\]

Consequently,

\[
w(r_i) \geq \left( \frac{\alpha_1}{\alpha_3} \right)^{\frac{1}{p+1}} w'(r_i)^{\frac{2}{p+1}}.
\]

(30)

By Lemma 15 we know that \( w \) is convex on \([w_i, r_i]\); consequently,

\[
\left( \frac{\alpha_1}{\alpha_3} \right)^{\frac{1}{p+1}} w'(r_i)^{\frac{2}{p+1}} \geq \left( \frac{\alpha_1}{\alpha_3} \right)^{\frac{1}{p+1}} w'(z_i)^{\frac{2}{p+1}}.
\]

(31)
Lastly, Lemma 16 gives us
\[
\left( \frac{\alpha_1}{\alpha_3} \right)^{\frac{1}{p+1}} \frac{1}{w'(z_i)} \geq \left( \frac{\alpha_1}{\alpha_3} \right)^{\frac{1}{p+1}} \left( \frac{C_1}{\alpha_2} \right)^{\frac{1}{p+1}} \left( \frac{4}{3} \right)^{\frac{(p+3)}{2(p+1)p}}.
\]
Combining (29), (30), and (31) results in
\[
w(r_i) \geq \left( \frac{\alpha_1 C_1}{\alpha_3 \alpha_2} \right)^{\frac{1}{p+1}} \left( \frac{4}{3} \right)^{\frac{(p+3)}{2(p+1)p}} i.
\]
A similar proof works if \( r_i \) is on an interval of negativity. \( \square \)

6. Estimating distances between zeros

In this section we will show the distances between consecutive zeros of \( w \) are bounded above by a summable geometric series. This observation along with the fact
\[
\lim_{i \to \infty} |w(m_i)| = \infty
\]
from Lemma 14 will give finite time blow-up for solutions of (2) provided the assumptions of Theorem 1 are satisfied.

Begin with the following auxiliary result for \( G \) similar to the one used back in Lemma 12 about oscillations of \( w \) for \( k \leq 0 \).

**Lemma 18.** Assume \( w \) is a global solution to Eq. (2), \( f \) satisfies (3) and (4), and the initial conditions satisfy (16). Let \( \lambda > 0 \) be such that
\[
\frac{(p+1)(1+\lambda)}{p-\lambda} < 2 \quad \text{and} \quad 2(1+\lambda) < p + 1
\] (\( \lambda \) exists since \( p > 1 \)). The set of points \( t \) in \([0, \infty)\) where \( w \) exists and
\[
|w'|^{2(1+\lambda)}(t) < 2w(t)f(w(t))
\]
has finite measure (the factor 2 is just for convenience).

**Proof.** From the definition of \( G \) it follows that whenever \( |w'|^{2(1+\lambda)} < 2wf(w) \) we have
\[
G^{1+\lambda} \lesssim |(w')^{2(1+\lambda)} + |w|^{2(1+\lambda)} + |ww'|^{1+\lambda} + 1
\]
\[
\lesssim |w'|^{2(1+\lambda)} + |w|^{2(1+\lambda)} + |w|^{p+1} + |w''|^{\frac{(p+1)(1+\lambda)}{p-\lambda}} + 1
\]
(Young’s Inequality)
\[
\lesssim |w'|^{2(1+\lambda)} + |w|^{p+1} + |w''|^2 + 1 \quad \text{(recall \( p + 1 \geq 2(1+\lambda) \))}
\]
\[
\lesssim wf(w) + |w''|^2 + 1 \quad \text{(condition (4) and \( |w'|^{2(1+\lambda)} < 2wf(w) \))}.
\]
From Lemma 5 we have
\[ G'' \geq \mu_1 (w')^2 + \mu_2 (w'')^2 + \mu_3 w f(w) + \varepsilon \]
\[ \geq \mu_1 (w')^2 + \mu_2 (w'')^2 + \mu_3 \rho |w|^{p+1} + \varepsilon \]
and so for some constant \( \beta > 0 \) we have
\[ \beta G''(t) > G(t)^{1+\lambda} \]
(33)
on sets where
\[ |w'(t)|^{2(1+\lambda)} < 2w(t)f(w). \]
Let \( U \) be the union of all (open) sets where \( |w'(t)|^{2(1+\lambda)} < 2w(t)f(w(t)) \), then on \( U \) the estimate (33) holds. Any solution \( u \) to the differential inequalities \( u' > \varepsilon > 0 \) and \( \beta u''(t) > u(t)^{1+\lambda} \) has only finite existence time. For \( T \) sufficiently large we know that \( G' > \varepsilon > 0 \) on \([T, \infty)\) by Lemma 5. Since \( G \) satisfies (33) on the set \( U \), and \( G \) remains strictly increasing outside \( U \), we conclude that \( |U| < \infty \), otherwise, \( G \) blows up in finite time. \( \square \)

**Remark 2.** The proof for Theorem 1 simplifies drastically in both cases, \( k > 0 \) and \( k \leq 0 \), if one shows inequality (33) holds on the entire real line. This inequality reduces to showing
\[ (w')^{2(1+\lambda)} \lesssim (w')^2 + (w'')^2 + w f(w). \]
Numerical evidence supports this conjecture, however, we were unable to prove it at this point.

The following lemma was motivated by [10, pp. 22–24] and will show
\[ \sum |m_i - z_{i+1}| < \infty. \]

**Lemma 19.** Assume \( w \) is a global solution to Eq. (2), \( f \) satisfies conditions (3) and (4), and the initial values satisfy (16). There exists a \( T \) sufficiently large such that if \( z_0 \in \mathcal{Z} \) and \( z_0 \geq T \) then
\[ |m_i - z_{i+1}| < \frac{C}{\sqrt{w(m_i)}} \leq C \left( \frac{3}{4} \right)^{\frac{p-1}{2(p+1)}} i \]
where \( i \in \mathbb{N} \) and \( C = C(k, \rho, p) \).

**Proof.** Let \( T \) be large enough for the consequences of Lemmas 14 and 15 to be valid and Lemma 5 implies \( G'(t) > 0 \) for \( t \geq T \). For convenience reindex the set of zeros \( \mathcal{Z} \) so that \( z_0 \geq T \). Since \( w(t - m_i) \) is a solution to Eq. (2) if and only if \( w(t) \) is, we may assume \( m_i = 0 \). Integrating \( w''' + kw'' + f(w) = 0 \) four times with respect to \( t \) results in
\[ w(t) = t^3 \frac{w'''(0)}{6} + t^2 \frac{w''(0)}{2} + \left( t + \frac{t^3 k}{6} \right) w'(0) + \left( 1 + \frac{t^2 k}{2} \right) w(0) \]

\[-k \int_0^t \int_0^{t_3} \int_0^{t_3} w(t_2) dt_2 dt_3 - \int_0^t \int_0^{t_3} \int_0^{t_2} f(w(t_0)) dt_0 dt_1 dt_2 dt_3.\] (34)

Suppose that \([z_i, z_{i+1}]\) is an interval of positivity of \(w\). Since \(m_i = 0\) is a maximum we know that \(w'(0) = 0\) and \(w''(0) \leq 0\). Additionally, by Lemma 5 we know \(G'(0) = -\alpha_3 w(0) w'''(0) \geq 0\). It follows that \(w'''(0) \leq 0\). Applying this to Eq. (34) yields

\[ w(t) \leq \left( 1 + \frac{t^2 k}{2} \right) w(0) - k \int_0^t \int_0^{t_3} \int_0^{t_3} w(t_2) dt_2 dt_3 - \int_0^t \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} f(w(t_0)) dt_0 dt_1 dt_2 dt_3.\]

Recall from condition (4), that \(f(w) \geq \rho w^p\) since we are on an interval of positivity. Thus

\[ w(t) \leq \left( 1 + \frac{t^2 |k|}{2} \right) w(0) + |k| \int_0^t \int_0^{t_3} \int_0^{t_3} w(t_2) dt_2 dt_3 - \rho \int_0^t \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} w(t_0)^p dt_0 dt_1 dt_2 dt_3.\]

By Lemmas 13 and 15 we know \(w\) is decreasing and concave on \([0, z_{i+1}]\). Consequently, for any \(\delta \in (0, z_{i+1})\) and \(t \in [0, z_{i+1}]\) we know that

\[ w(t) \geq \ell(t) := w(0) - \frac{w(0)}{\delta} t \]

and \(w(t) \leq w(0)\). We conclude

\[ w(t) \leq \left( 1 + \frac{t^2 |k|}{2} \right) w(0) + |k| w(0) \int_0^t \int_0^{t_3} dt_2 dt_3 - \rho \int_0^t \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} \rho \ell(t_0)^p dt_0 dt_1 dt_2 dt_3.\]

Evaluating at \(t = \delta\) gives us

\[ w(\delta) \leq \frac{w(0)}{24 + 6p} (24 + 6p + \delta^2(6|k|p + 24|k|) - w(0)^{p-1} \rho \delta^4).\]

Since \(w(\delta) \geq 0\), we have

\[ 0 \leq 24 + 6p + \delta^2(6|k|p + 24|k|) - w(0)^{p-1} \rho \delta^4.\]

Consequently,

\[ \delta^2 \leq \frac{(6|k|p + 24|k|) + \sqrt{(6|k|p + 24|k|)^2 + 4\rho (24 + 6p) w(0)^{p-1}}}{2 \rho w(0)^{p-1}}.\]
This inequality along with Lemma 14 (about $w(m_i)$ being eventually large, whence here we may assume that for $w(0)$) implies that

$$\delta \leq \frac{C}{w(0)^{\frac{p-1}{4}}},$$

for some $C = C(k, p, \rho) > 0$.

Note that $C$ is not dependent on $m_i$. This estimate holds uniformly for every $\delta \in [m_i, z_{i+1})$, hence

$$|m_i - z_{i+1}| \leq \frac{C}{w(0)^{\frac{p-1}{4}}} = \frac{C}{w(m_i)^{\frac{p-1}{4}}}.$$

The result follows by recalling $|w(m_i)| \geq \left(\frac{4}{3}\right)^\frac{p+1}{2}$ from Lemma 14. A similar proof works for an interval of negativity. □

The next and final lemma will show that $\sum |z_i - m_i| < \infty$.

**Lemma 20.** Assume $w$ is a global solution to Eq. (2), $f$ satisfies conditions (3) and (4), and the initial values satisfy (16). There exists a $T \geq 0$ such that if $z_i \geq T$, then

$$\sum_{i=1}^{\infty} |z_i - m_i| < \infty.$$

**Proof.** Let $T$ be sufficiently large so that $G' > 0$ for $t \geq T$ (Lemma 5) and the consequences of Lemmas 13, 16, and 17 are valid. For convenience reindex the set of zeros $Z$ so that $z_0 \geq T$. Recall $w(t) f(w(t)) \geq \rho |w|^{p+1}$ by condition (4). Let $\lambda > 0$ be chosen as in Lemma 18. By Lemma 18 we know the total measure of sets where

$$|w'(t)|^{2(1+\lambda)} < 2w(t) f(w(t))$$

is finite so we will consider subsets of $[z_i, m_i]$ where

$$|w'(t)|^{2(1+\lambda)} > w(t) f(w(t)) \geq \rho |w(t)|^{p+1}.$$

The sets where these (strict) inequalities hold forms an open cover of the interval of existence of $w$. Let us focus on a subinterval $[z_i, m_i]$ and assume it resides in an interval of positivity.

We know the inequality $|w'|^{2(1+\lambda)} > \rho |w|^{p+1}$ is equivalent to

$$w' > \rho^{\frac{1}{2(1+\lambda)}} w^{\frac{p+1}{2(1+\lambda)}}$$

since $w, w'$ are nonnegative.

By Lemma 17 we know that $\xi_i \leq r_i$ where $r_i$ is the first inflection point on $[z_i, m_i]$. Using the estimate on $w(r_i)$ from Lemma 17 we see that there exists $\xi_i \in (z_i, r_i]$ such that

$$w(\xi_i) = v := \left(\frac{\alpha_1 C_1}{\alpha_2 \alpha_3}\right)^{\frac{1}{p+1}} \left(\frac{4}{3}\right)^{\frac{(p+3)}{2(p+1)^2}}.$$
We will now give a bound on the measure of sets contained in $[\xi_i, m_i]$ where $w' > \rho^{\frac{1}{2(1+\lambda)}} w^{\frac{p+1}{2(1+\lambda)}}$. Consider the initial value problem

\[
\begin{cases}
    u' = \rho^{\frac{1}{2(1+\lambda)}} u^{v+1} \\
u := \frac{p+1}{2(1+\lambda)} - 1 > 0.
\end{cases}
\]

Solving the above equation for $u$ results in

\[
u(t) = \left( -u^{\rho^{\frac{1}{2(1+\lambda)}}} + \nu \rho^{\frac{1}{2(1+\lambda)}} \xi_i + \frac{1}{u_{\xi_i}} \right)^{-\frac{1}{v}} = \left( \nu \rho^{\frac{1}{2(1+\lambda)}} (\xi_i - t) + \frac{1}{u_{\xi_i}} \right)^{-\frac{1}{v}}.
\]

It is clear that the solution $u$ to the above initial value problem is only defined for $t < \xi_i + (u_{\xi_i} \nu \rho^{\frac{1}{2(1+\lambda)}})^{-1}$. Thus, using the definition of $u_{\xi_i}$, we see that the existence time for $u$ in the interval $[\xi_i, m_i]$ is no more than

\[
\left( u_{\xi_i} \nu \rho^{\frac{1}{2(1+\lambda)}} \right)^{-1} \leq C(\alpha_1, \alpha_2, \phi, \rho, k) \left( \frac{4}{3} \right)^{-\frac{\phi(p+3)}{2(p+1)^2}} \text{ with } \phi > 0.
\]

By Lemma 11 we know $w$ monotonically increases to $w(m_i)$ on the first part $[z_i, m_i]$ of the positivity interval $[z_i, z_{i+1}]$. Thus the existence time in $[\xi_i, m_i]$ for $w$ where $|w'|^{2(1+\lambda)} > \rho |w|^{p+1}$ is no larger than the existence time for $u$; therefore, an upper bound on the measure of sets where $|w'|^{2(1+\lambda)} > \rho |w|^{p+1}$ in $[\xi_i, m_i]$ is given by $C(\alpha_1, \alpha_2, \phi, \rho, k) \left( \frac{4}{3} \right)^{-\frac{\phi(p+3)}{2(p+1)^2}}$. We conclude

\[
\sum_{i=1}^{\infty} |\xi_i - m_i| < \infty.
\]

Now let us consider the interval $[z_i, \xi_i]$. Since $\xi_i \leq r_i$ we know that $w$ is convex on the interval. Thus $w(\xi_i) \geq (\xi_i - z_i) w'(z_j)$. Appealing to Lemma 16, we obtain

\[
|\xi_i - z_i| < w(\xi_i) (w'(z_j))^{-1} \leq \left( \frac{\alpha_1 C_1}{\alpha_2 \alpha_3} \right)^{\frac{1}{p+1}} \left( \frac{4}{3} \right)^{\frac{(p+3)i}{2(p+1)^2}} \left( \frac{\alpha_2}{C_1} \right)^{\frac{1}{2}} \left( \frac{3}{4} \right)^{\frac{(p+3)i}{4(p+1)}} \left( \frac{\alpha_1 C_1}{\alpha_2 \alpha_3} \right)^{\frac{1}{p+1}} \left( \frac{4}{3} \right)^{\frac{(p+3)i}{4(p+1)^2}} \left( \frac{\alpha_1 C_1}{\alpha_2 \alpha_3} \right)^{\frac{1}{2}} \left( \frac{3}{4} \right)^{\frac{(p+3)i}{4(p+1)}}.
\]
Since $p > 1$ we know $\frac{p+3}{4(p+1)} - \frac{p+3}{2(p+1)^2} > 0$ and so

$$
\sum_{i=1}^{\infty} |\xi_i - z_i| < \infty.
\quad \Box
$$

Now we are in position to finish the proof of Theorem 1.

**Proof of Theorem 1.** We will show that no global solution exists. To the contrary, assume $w$ is a global solution to (2) with initial energy satisfying (5). By Theorem 10 and Lemma 12 we know that $w$ must change sign infinitely many times as $t \to \infty$. If $\{m_i\}$ are the extrema of $w$, Lemma 14 implies $|w(m_i)| \to \infty$ as $i \to \infty$. Lemmas 19 and 20 imply the zeros of $w$, hence $m_i$’s, have a limit point $m_\infty$. We conclude $w(t)$ is unbounded in a neighborhood of $m_\infty$. So $w$ cannot be extended past $m_\infty$. \&

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**Appendix A. Oscillations of $w$ when $k \leq 0$**

**Lemma 12.** Assume $f$ satisfies conditions (3) and (4), $k \leq 0$, and $w$ is a global solution of (2) with initial conditions satisfying (16). Then $w$ is never eventually of one sign.

**Proof.** We will assume that $w$ is eventually nonnegative. A similar proof will work in the case when $w$ is eventually nonpositive. Suppose there exists a $T \geq 0$ such that $w(t) \geq 0$ for $t \geq T$. We will show that $w$ blows up in finite time. Let $T$ be sufficiently large so the consequences of Lemma 11 hold.

**Step 1. We will show that $w' \geq 0$ for $t \geq T$.** Suppose $w' < 0$ for some $t \geq T$. Lemma 11 implies $w' < 0$ as long as $w \geq 0$; thus $w' < 0$ for all $t \geq T$. Recall

$$
G' = 2\alpha_1 w w' + (2\alpha_2 - \alpha_3) w'' w' - \alpha_3 w w''
$$

$$
E = \frac{k}{2} (w')^2 + w w''' + F(w) - \frac{\alpha_3}{2} (w'')^2.
$$

From Lemma 3 we know $E(t)$ is constant. From Lemma 5 we know $G' \to \infty$ (monotonically for $t$ large) as $t \to \infty$. We conclude

$$
G' + \alpha_3 E = 2\alpha_1 w w' + (2\alpha_2 - \alpha_3) w'' w' + \frac{k\alpha_3}{2} (w')^2 + \alpha_3 F(w) - \frac{\alpha_3}{2} (w'')^2 \to \infty
$$

(monotonically for $t$ large) as $t \to \infty$.

Since $w$ is bounded, we know $F(w)$ is bounded. Thus
\[2 \alpha_1 w w' + (2 \alpha_2 - \alpha_3) w' w'' + \frac{k \alpha_3}{2} (w')^2 - \frac{1}{2} (w'')^2 \rightarrow \infty \quad \text{as } t \rightarrow \infty.\]

Recall that \(w \geq 0, w' < 0, k \leq 0\), consequently, dropping nonpositive terms gives

\[(2 \alpha_2 - \alpha_3) w' w'' \rightarrow \infty.\]

Since \(\alpha_2 > \alpha_3\) by (9) and \(w' < 0\), we must have \(w'\) or \(w'' \rightarrow -\infty\) contradicting \(w \geq 0\).

We conclude

\[w, w' \geq 0 \quad \text{for } t \in (T, \infty). \quad (A.1)\]

**Step 2. We claim that** \(w \rightarrow \infty\). **Since** \(w\) is monotone increasing \((w' \geq 0)\), **it suffices to prove** that \(w\) is unbounded. **To the contrary, assume it is bounded.** From (2) and (4)

\[\frac{d}{dt}(w''' + k w') = w''' + kw' = -f(w) \leq -\rho w^p\]

it follows that for \(t\) large \(w''' + k w'\) is strictly decreasing. **Thus** we may assume that \(T\) is, in addition, sufficiently large so that for some constant \(C\)

\[w''' < C - kw', \quad t \geq T. \quad (A.2)\]

Since \(w \geq 0\) we have from the definition of the energy

\[E \leq \frac{k}{2} (w')^2 + w(C - kw') + F(w) - \frac{1}{2} (w'')^2.\]

By Lemma 5 we have \(\lim_{t \to \infty} (G + E) = \infty\) so,

\[\alpha_1 w^2 + \alpha_2 (w')^2 - \alpha_3 w w'' + \frac{k \alpha_3}{2} (w')^2 + w(C - kw') + F(w) - \frac{1}{2} (w'')^2 \geq G + E \rightarrow \infty \quad \text{as } t \rightarrow \infty.\]

By our assumption that \(w\) is bounded, we have \(\alpha_1 w^2, Cw, F(w)\), and \(-\alpha_3 w w'' - \frac{1}{2} (w'')^2\) are all bounded above. Dropping these terms and the non-positive quantity \(\frac{k}{2} (w')^2\) (note that \(k \leq 0\)) we arrive at

\[\lim_{t \to \infty} (\alpha_2 (w')^2 + |k| w w') = \infty.\]

Since \(w \geq 0\) is bounded and \(w' \geq 0\) we must have \(\lim_{t \to \infty} w' = \infty\) contradicting \(w\) being bounded. Thus

\[w \rightarrow \infty \quad \text{monotonically on } (T, \infty). \quad (A.3)\]

**Step 3. Positivity of** \(w''\). **It follows from** (4) and (A.3) that

\[-f(w) \rightarrow -\infty \quad \text{as } t \rightarrow \infty.\]
Now let us look at the behavior of \( w''(t) \). We will show that \( w''(t) \geq 0 \) for all \( t \) sufficiently large. To the contrary, suppose \( w''(t) \leq 0 \) for all \( t \in (T_1, \infty) \) for some \( T_1 \geq T \). From Eq. (2) and the assumption \( k \leq 0 \) we get

\[
\frac{d^2}{dt^2}(kw + w'') = kw'' + w'''' = -f(w) \\
\leq -\rho w^p 
\]

which implies that \( w(t) \to -\infty \) as \( t \to \infty \), contradicting our initial assertion that \( w \geq 0 \).

We conclude that \( w'' \) must be positive somewhere on \( (T_1, \infty) \) and strictly concave on \( (b_1, b_2) \), whence \( b_2 = \infty \). But then we contradict the earlier observation that \( w'' \) cannot remain negative on an interval of the form \( (T_1, \infty) \).

Thus, for \( T \) sufficiently large,

\[
w, w', w'' \geq 0, \quad t \in (T, \infty). \tag{A.4}
\]

**Step 4. Finite-time blow-up.** From Eq. (2) we have

\[
\frac{d^2}{dt^2}(kw + w'') = kw'' + w'''' = -f(w) \\
\leq -\rho w^p \quad \text{(condition (4) for \( |w| \) large).} \tag{A.5}
\]

Recall from (A.3) that \( w \to \infty \) monotonically. Combining this with (A.5) allows us to find a \( C > 0 \) such that for sufficiently large \( t \) we have

\[
\frac{d^2}{dt^2}(kw(t) + w''(t)) \leq -C < 0.
\]

So for all large \( t \), \( kw + w'' < 0 \). In other words, we may assume that \( T \) is large enough so that

\(-kw = |k|w > w'' \) on \( (T, \infty) \).

To summarize: on \( (T, \infty) \) we have \( w, w', w'' \geq 0 \) and \( |k|w > w'' \). Consequently

\[
G' = 2\alpha_1 w w' + 2\alpha_2 w' w'' - \alpha_3 (w' w'' + w w''') \\
\leq (\alpha_1 + \alpha_2 |k|) 2w' w - \alpha_3 (w' w'' + w w''').
\]

For \( t \geq T \) we obtain:

\[
G(t) - G(T) = \int_T^t G' ds \\
\leq (\alpha_1 + \alpha_2 |k|) w(t)^2 - \alpha_3 w(t)w''(t) + C_T. \tag{A.6}
\]
Pick $\lambda > 0$ such that $\frac{(p+1)(1+\lambda)}{p-\lambda} < 2$ (equivalent to $\lambda < \frac{p-1}{p+3}$) and $2(1+\lambda) < p+1$. Since $p > 1$, such a $\lambda$ exists. With $\lesssim$ below indicating omitted positive constant factors (independent of $t$ or the solution) we have for $t \geq T$,

$$G^{1+\lambda} \lesssim \left( (\alpha_1 + \alpha_2|k|)|w|^2 - \alpha_3 w w'' + C_T + G(T) \right)^{1+\lambda} \quad \text{(by (A.6))}$$

$$\lesssim \left( |w|^2 + |ww''| + 1 \right)^{1+\lambda}$$

$$\lesssim |w|^2(1+\lambda) + |ww''|^{1+\lambda} + 1$$

$$\lesssim |w|^p + |w''|^p(1+\lambda) + 1 \quad \text{(by Young’s inequality)}$$

$$\lesssim |w|^p + |w''|^p + 1 \quad \text{(recall } p+1 \geq 2(1+\lambda) \text{ and } \frac{(p+1)(1+\lambda)}{p-\lambda} < 2 \text{)}$$

$$\lesssim w f(w) + |w''|^2 + 1 \quad \text{(by condition (4))}$$

$$\lesssim G'' \quad \text{(Lemma 5).}$$

From Lemma 5 we know $G'' \geq \varepsilon > 0$, so $G$ is positive and strictly increasing for $t$ sufficiently large. This along with the fact $G^{1+\lambda} \lesssim G''$ for all $t \geq T$ implies $G$ blows up in finite time. This in turn implies $w$ blows up in finite time. \qed

References


