DIRICHLET'S PRINCIPLE AND WELLPOSEDNESS OF SOLUTIONS FOR A NONLOCAL $p-$LAPLACIAN SYSTEM WITH APPLICATIONS IN PERIDYNAMICS

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Abstract. We prove Dirichlet's principle for a nonlocal p-Laplacian system which arises in the nonlocal setting of peridynamics when $p = 2$. This nonlinear model includes boundary conditions imposed on a nonzero volume collar surrounding the domain. Our analysis uses nonlocal versions of integration by parts techniques that resemble the classical Green and Gauss identities. The nonlocal energy functional associated with this "elliptic" type system exhibits a general kernel which could be weakly singular. The coercivity of the system is shown by employing a nonlocal Poincaré's inequality. We use the direct method in calculus of variations to show existence and uniqueness of minimizers for the nonlocal energy, from which we obtain the wellposedness of this steady state diffusion system.

1. Introduction

In classical continuum mechanics materials are assumed to be continuously distributed entirely throughout the space they occupy. However, in some materials cracks and other discontinuities arise, hindering the use of the classical model, as the governing equations collapse at singularities. As a result, a nonlocal theory called peridynamics was introduced in 2000 by Silling [24] who prescribes that spatial derivatives may be avoided in modeling the interactions between particles. In this nonlocal version of continuum theory, internal forces acting on material particles are assumed to form a network of pairwise forces, known as bonds. Each material point interacts with all the neighboring points within a given region surrounding the point; this region is called the horizon. In peridynamics a fracture is seen as a breaking of the aforementioned pairwise bonds, which can be modeled by a vanishing pairwise force function. Thus, the strength of the peridynamic formulation is that the same equations can be applied to all points in the domain, eliminating the previous need for special techniques from fracture mechanics. The peridynamic theory has successfully dealt with with materials that are homogeneous [10, 24, 23] and also with materials that are heterogeneous [4] by replacing the partial differential operators found in classical continuum theory with integral or integro-differential operators. The theory of peridynamics has received considerable attention from the engineering community, however, the mathematical theory providing rigorous proofs for qualitative and quantitative properties of these systems is still in its early development stages.

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In this paper we consider the following nonlocal "elliptic" boundary value problem

$$\begin{cases} \mathcal{L}_p u(x) = b(x), & x \in \Omega \\ u(x) = g(x), & x \in \Gamma, \end{cases} \tag{1.1}$$

where

$$\mathcal{L}_p u(x) := 2 \int_{\Omega\cup\Gamma} |u(x') - u(x)|^{p-2}(u(x') - u(x))\mu(x, x')\, dx'. \tag{1.2}$$

Here Ω denotes an open bounded subset of $\mathbb{R}^n$, $\Gamma \subset \mathbb{R}^n\backslash\Omega$ denotes a "collar" domain surrounding Ω which has nonzero volume, and $1 < p < \infty$. The nonhomogeneities b and g are L^2 integrable functions over Ω, respectively Γ. The kernel $\mu(x, x')$ denotes a positive, symmetric function of its arguments, with a singularity around $x = x'$ which records the interaction of $x \in \Omega$ with its neighboring points x' from the horizon of x. To account for the interaction of x with points outside the domain Ω, when $x \in \partial\Omega$, we allow x' to belong to the collar domain Γ which contains the horizons $\mathcal{H}_x$ as x moves along $\partial\Omega$. For $p = 2$ the operator $\mathcal{L}_p$ is the operator considered in peridynamic models of elasticity and heat diffusion [10, 24]. This operator is also a nonlocal extension of the classical p-Laplacian $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, or the operator from the porous medium equation $\Delta(|u|^{m-1}u)$. An in-depth discussion of the evolution equation associated with this nonlocal operator may be found in Chapter 6 of the monograph [9].

The use of nonlocal operators such as $\mathcal{L}_p$ has been proven valuable in applications from several areas, including image processing [17, 18, 19], sandpile formation [7], swarm [22] and other population density models [13, 12]. Current literature from the last decade contains an in depth investigation of nonlocal systems such as (1.1), but usually this is performed under smoothness assumptions for the kernel (see for example [5, 6, 8, 9] and the references therein).

The system (1.1) is encountered in peridynamic models when $p = 2$ with a prototype kernel $\mu(x, x')$ of the form

$$\mu(x, x') = \begin{cases} \frac{1}{|x-x'|^{\beta}} & \text{for } |x - x'| < \delta \\ 0 & \text{for } |x - x'| \geq \delta, \end{cases} \tag{1.3}$$

where $\beta > 0$. In the linear case (when $p = 2$), with the above choice for μ, for $\beta > n$ the natural framework to study regularity properties of the operator $\mathcal{L}_p$ is that provided by fractional derivative spaces (see [1]). For $\beta < n$ the kernel is *weakly singular* and the derivation of classical regularity results can not be done following standard techniques. Some of the difficulties arising in this situation stem from the fact that the operator is *non-smoothing*; also, there is no higher integrability for the function based on the L^2 bounds of the nonlocal gradient as given by the nonlocal version of Poincaré's inequality (see Lemma 3.5 and Remark 3.7).

An important aspect in the development of the peridynamic theory is the consideration of boundary conditions. Due to the nonlocal property of the operator $\mathcal{L}_p$, classical boundary conditions (imposed on boundaries of zero volume) will not yield well posed systems. Gunzburger and Lehoucq resolved this issue in [20] where they define weak formulations for the nonlocal boundary value problem (1.1) in the scalar case for $p = 2$ and show wellposedness of these problems after developing a calculus for nonlocal operators. A rigorous framework for nonlocal calculus of vectors and tensors was recently developed in [15].

The nonlocal $p-$Laplacian operator defined here is an extension of the operator $\mathcal{L}$ from [20], that corresponds to the case $p = 2$. In [20], however, the integral is considered over the domain Ω alone. We chose to employ the integral operator over $\Omega \cup \Gamma$ because the nonlocal equations associated with this operator have the form of natural generalizations of the classical PDEs, and the identities presented throughout the paper are simplified. Indeed, in Lemma 2.1 and Proposition 2.5 we verify properties of $\mathcal{L}_p$ that are nonlocal analogues of properties of the classical Laplacian. Gunzburger and Lehoucq have shown in [20] that for $p = 2$ the problem (1.1) provides a nonlocal equivalent of the classical elliptic boundary value problem

$$\begin{cases} \Delta u(x) = b(x), & x \in \Omega \\ u(x) = g(x), & x \in \Gamma. \end{cases}$$

More specifically, they show that the solution of $\mathcal{L}u = \delta(x' - x)$ becomes the fundamental solution of the Laplacian when one chooses $\mu = \delta(x'-x) + \frac{\partial^2}{\partial x^2}\delta(x'-x)$. In addition, [20] provides wellposedness results of the system (1.1), when $p = 2$ and μ is weakly singular, by using a variational approach based on the Lax-Milgram lemma.

The Lax-Milgram variational approach is not applicable for our nonlinear system, so we use the direct method in calculus of variations to show existence and uniqueness of solutions. Our first contribution is the proof of Dirichlet's principle in the given nonlocal setting. More precisely, we show that the minimizers of the energy functional

$$\mathcal{F}[u] = \frac{1}{2}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} |u(x') - u(x)|^p \mu(x, x')dx'dx + \int_{\Omega} b(x)u(x)dx, \tag{1.4}$$

satisfy (1.1) and conversely, any solution of (1.1) is a minimizer for $\mathcal{F}$. This result then enables us to prove the wellposedness of the system (1.1) by showing existence and uniqueness of minimizers. As in the classical setting (see [14]) the existence of minimizers relies on convexity and coercivity properties of the integrand. For our functional, the convexity is immediate due to the structure of our integrand, and this ensures the necessary weakly lower semicontinuity. The coercivity property requires the use of a nonlocal Poincaré type inequality which is proven in [2]. Since our proof follows the direct method in calculus of variations it will not need the completeness assumptions on the space of functions, therefore, our kernels could be chosen in great generality. Thus, we do not rely on the existence of a uniform radius δ for the horizon in the peridynamic representation (see Remark 2.3). Also, the kernel function $\mu = \mu(x, x')$ does not need to be solely a function of $|x - x'|$, as it only needs to satisfy the assumption (A1) with the bound (2.8). The existence of minimizers was proven in [16] for the problem on $\mathbb{R}^n$, but under growth restrictions on the growth of the kernel. Our results hold for *any* β when the domain Ω is bounded.

The paper is organized as follows: Section 2 contains a discussion of notation, definitions, and elliptic-type properties involving the operator $\mathcal{L}_p$. Also, in Section 2 we collect and prove some nonlocal calculus results developed in [8] and [20] which provide nonlocal analogues of both Gauss's Theorem and Green's identities. In Section 3, we define the nonlocal energy functional associated with (1.1) and prove the Dirichlet's Principle in the nonlocal setting. Subsection 3.2 contains the proof for existence and uniqueness of minimizers based on the nonlocal Poincaré's inequality given in Lemma 3.5. These results yield the well-posedness of the system (1.1).

2. Preliminaries

For $\alpha : (\Omega \cup \Gamma) \times (\Omega \cup \Gamma) \to \mathbb{R}^n$, $u : \Omega \cup \Gamma \to \mathbb{R}$, and $f : (\Omega \cup \Gamma) \times (\Omega \cup \Gamma) \to \mathbb{R}^n$ we define as in [20] the following generalized nonlocal operators:
(i) *Generalized gradient*

$$\mathcal{G}(u)(x, x') := (u(x') - u(x))\alpha(x, x'), \quad x, x' \in \Omega \cup \Gamma, \tag{2.1}$$

(ii) *Generalized nonlocal divergence*

$$\mathcal{D}(f)(x) := \int_{\Omega \cup \Gamma} (f(x, x') \cdot \alpha(x, x') - f(x', x) \cdot \alpha(x', x))dx', \quad x \in \Omega, \tag{2.2}$$

(iii) *Generalized normal component*

$$\mathcal{N}(f)(x) := -\int_{\Omega \cup \Gamma} (f(x, x') \cdot \alpha(x, x') - f(x', x) \cdot \alpha(x', x))dx', \quad x \in \Gamma. \tag{2.3}$$

With the above notation in place, the authors in [20] demonstrate that for $v : \Omega \cup \Gamma \to \mathbb{R}$ and $s : (\Omega \cup \Gamma) \times (\Omega \cup \Gamma) \to \mathbb{R}^n$ the following identity holds

$$\int_\Omega v\mathcal{D}(s)dx + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} s \cdot \mathcal{G}(v)dx'dx = \int_\Gamma v\mathcal{N}(s)dx \tag{2.4}$$

Note that by choosing $v(x)$ to be a constant, we arrive at the following nonlocal Gauss's theorem:

$$\int_\Omega \mathcal{D}(f)dx = \int_\Gamma \mathcal{N}(f)dx.$$

Its connection to the classical Gauss's theorem is shown in [20].

For the remainder of the paper we let $\mu : (\Omega \cup \Gamma) \times (\Omega \cup \Gamma) \to \mathbb{R}$ be given by

$$\mu(x, x') := |\alpha(x, x')|^p.$$

We will assume throughout the paper that α is symmetric (i.e. we have $\alpha(x, x') = \alpha(x', x)$, for every $x, x' \in \Omega \cup \Gamma$), hence μ is symmetric as well. With this choice for μ the following identity :

$$\mathcal{D}(|\mathcal{G}(u)|^{p-2}\mathcal{G}(u)) = 2\int_{\Omega \cup \Gamma} |(u(x') - u(x)|^{p-2}(u(x') - u(x))\mu(x, x')dx', \tag{2.5}$$

was shown in [20], eq.(5.3) for $p = 2$. We state and prove the general case below.

Lemma 2.1. *For every measurable function $u : \Omega \cup \Gamma \to \mathbb{R}$ we have the following identity*

$$\mathcal{L}_p u = \mathcal{D}(|\mathcal{G}(u)|^{p-2}\mathcal{G}(u)), \tag{2.6}$$

where $\mathcal{L}_p, \mathcal{G}$, and $\mathcal{D}$ are defined in (1.1), (2.1), and (2.2). The equality holds whenever both sides are finite.

Proof. We have the following identities

$$\begin{aligned}
&\mathcal{D}(|\mathcal{G}(u)|^{p-2}\mathcal{G}(u))(x)\\
= &\int_{\Omega\cup\Gamma} |\mathcal{G}(u)(x)|^{p-2}\mathcal{G}(u)(x)\alpha(x,x') - |\mathcal{G}(u)(x')|^{p-2}\mathcal{G}(u)(x')\alpha(x',x)dx'\\
= &\int_{\Omega\cup\Gamma} (u(x')-u(x))\alpha^2(x,x')(|\mathcal{G}(u)(x)|^{p-2} + |\mathcal{G}(u)(x')|^{p-2})dx'\\
= &\ 2\int_{\Omega\cup\Gamma} (u(x')-u(x))\alpha^2(x,x')|u(x')-u(x)|^{p-2}|\alpha(x,x')|^{p-2}dx'\\
= &\ 2\int_{\Omega\cup\Gamma} |u(x')-u(x)|^{p-2}|\alpha(x,x')|^{p}(u(x')-u(x))dx'\\
= &\ 2\int_{\Omega\cup\Gamma} |u(x')-u(x)|^{p-2}(u(x')-u(x))\mu(x,x')dx'\\
= &\ \mathcal{L}_p u(x).
\end{aligned}$$

Hence, $\mathcal{L}_p u = \mathcal{D}(|\mathcal{G}(u)|^{p-2}\mathcal{G}(u))$, the desired result, a generalization of the classical identity $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \Delta_p u$.

□

The following lemma provides an integration by parts type identity; it can also be found in [9] (Lemma 6.5 on page 125); we provide a proof for completeness below. Note that in contrast with the local theory, this nonlocal version of the integration by parts formula does not contain any boundary terms, hence, the values of u on the domain Γ could be nonzero without affecting the equality.

Lemma 2.2. *Let $u, v : \Omega\cup\Gamma \to \mathbb{R}$ be measurable. Then one has the following "integration by parts" identity:*

$$\int_{\Omega\cup\Gamma} (\mathcal{L}_p u)v\,dx = -\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} |\mathcal{G}(u)|^{p-2}\mathcal{G}(u)\cdot\mathcal{G}(v)dx'dx. \tag{2.7}$$

Proof. Applying (2.4) with $s(x,x') = |\mathcal{G}(u)|^{p-2}\mathcal{G}(u)(x,x')$ and using the result in Lemma 2.1 we obtain

$$\int_{\Omega}(\mathcal{L}_p u)vdx = -\int_{\Gamma}(\mathcal{L}_p u)vdx - \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} |\mathcal{G}(u)|^{p-2}\mathcal{G}(u)\cdot\mathcal{G}(v)dx'dx.$$

Thus

$$\int_{\Omega\cup\Gamma}(\mathcal{L}_p u)vdx = -\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} |\mathcal{G}(u)|^{p-2}\mathcal{G}(u)\cdot\mathcal{G}(v)dx'dx,$$

the desired result. □

2.1. **Assumptions and properties of the nonlocal operator $\mathcal{L}_p$.** Throughout the paper we will work under the following assumption:

(A1). *Let $\mu : (\Omega\cup\Gamma)\times(\Omega\cup\Gamma) \to \mathbb{R}$ be a nonnegative measurable function. For every $x' \in \mathcal{H}_x$, there exists a constant $C_0 > 0$ such that $\mu(x,x') \geq C_0$. In other words, for all $x \in \Omega\cup\Gamma$ we have:*

$$C_0\chi_\delta(x,x') \leq \mu(x,x'), \tag{2.8}$$

where $\chi_\delta(x,x') = \begin{cases} 1, & |x-x'| \le \delta \\ 0, & |x-x'| > \delta. \end{cases}$

(A2). The domain Ω is an open bounded domain in $\mathbb{R}^n$ and Γ is a collar surrounding Ω, i.e. $\Omega \cap \Gamma = \partial\Omega$. Also, from a physical point of view we impose that Γ contains all the horizons $\mathcal{H}_x$; mathematically, in light of assumption (A1) above, we only need that

$$\cup_{x\in\partial\Omega} B_\delta(x) \subset \Gamma.$$

Remark 2.3. Note that the prototype function μ given in (1.3), satisfies this assumption. Also, our assumption allows for horizons which are not spherical; all we need is to find a ball centered at x (inside the horizon), where the kernel is bounded uniformly from below. This presents the following interesting aspect: *physically*, the interactions that the material experiences may take place on a large irregular domain, but *mathematically* all that is necessary is the existence of a small region where the interactions are non-zero.

Remark 2.4. In [20] the kernel μ is assumed to satisfy (as given by inequalities (6.1)):

$$\int_{\Omega\cup\Gamma} \mu(x,x')dx' \le C_1 \quad \text{and} \quad \int_{\Gamma} \mu(x,x')dx' \ge C_2.$$

These assumptions essentially impose that μ has the growth of a weakly singular operator; in other words, the exponent β in (1.3) must satisfy $\beta < n$ for all $x \in \Omega\cup\Gamma$. Here we only impose lower bounds through (2.8) which give that $\beta \ge 0$ for a kernel of a form found in (1.3).

We will prove next some properties of the nonlocal operator $\mathcal{L}_p$. These properties appear in [18] under a connectivity assumption which is satisfied, but we will not use it here (instead, we have (2.8) which ensures that $\mu \neq 0$ inside the horizon).

Proposition 2.5. *The operator $\mathcal{L}_p$ admits the following properties:*

(a) *If $u \equiv$ constant then $\mathcal{L}_p u = 0$.*
(b) *Let $x \in \Omega\cup\Gamma$. For any maximal point x_0 that satisfies $u(x_0) \ge u(x)$, we have $-\mathcal{L}_p u(x_0) \ge 0$. Similarly, if x_1 is a minimal point such that $u(x_1) \le u(x)$, then $-\mathcal{L}_p u(x_1) \le 0$.*
(c) *$\mathcal{L}_p u$ is a positive semidefinite operator, i.e. $\langle -\mathcal{L}_p u, u\rangle \ge 0$, where $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Omega\cup\Gamma)$ inner product.*
(d) $\displaystyle\int_{\Omega\cup\Gamma} -\mathcal{L}_p u(x)dx = 0.$

Remark 2.6. The converse of part (a), which states that the solution u of the system

$$\begin{cases} \mathcal{L}_p u = 0, & x \in \Omega \\ u = 0, & x \in \Gamma \end{cases}$$

must be identically zero follows from the wellposedness of the system, which is a result of Theorems 3.8 and 3.11.

Proof. Property (a) is trivial because if $u \equiv$ constant, then for all x and x' we have $u(x) - u(x') = 0$, hence $\mathcal{L}_p u = 0$.

For property (b), let x_0 be a maximal point. Then for all $x \in \Omega\cup\Gamma$ we have $u(x_0) \ge u(x)$. Then

$$-\mathcal{L}_p u(x_0) = 2\int_{\Omega\cup\Gamma} (u(x_0) - u(x'))|u(x_0) - u(x')|^{p-2}\mu(x_0,x')dx' \le 0$$

since $\mu(x_0, x') \geq 0$ by assumption (2.8) and $u(x_0) - u(x') \leq 0$. A similar argument can be used for a minimal point.

To obtain (c), note that we have by Lemma 2.2 that

$$\langle -\mathcal{L}_p u, u\rangle = \langle \mathcal{G}(u)|\mathcal{G}(u)|^{p-2}, \mathcal{G}(u)\rangle \geq 0$$

which shows that $\mathcal{L}_p u$ is a positive semidefinite operator.

For (d), we use again an integration by parts argument

$$\int_{\Omega\cup\Gamma} -\mathcal{L}_p u(x)dx = \langle -\mathcal{L}_p u, 1\rangle = \langle \mathcal{G}(u)|\mathcal{G}(u)|^{p-2}, \mathcal{G}(1)\rangle = 0$$

since $\mathcal{G}(1) = 0$. □

3. Well-Posedness of the Nonlocal Boundary-Value Problem

In this section we will first prove a nonlocal version of Dirichlet's Principle (Theorem 3.8). Next, we will establish the existence and uniqueness of minimizers of the functional $\mathcal{F}$ defined below. Combining these results, we obtain the well-posedness of the problem (1.1). First we will introduce the spaces and the framework in which we will present the results.

Let $\mu : (\Omega\cup\Gamma)^2 \to \mathbb{R}$ a nonnegative measurable function and consider its support given by the closure of the set of all the points where μ does not vanish:

$$N_\mu := \overline{\{(x, x') \in (\Omega\cup\Gamma)^2 | \mu(x, x') \neq 0\}} = supp(\mu).$$

Remark 3.1. Note that the assumption (A1) with the bound (2.8) implies that N_μ contains the band $\{(x, x') \in (\Omega\cup\Gamma)^2 | |x - x'| \leq \delta\}$. As a consequence of this fact, we have that $N_\mu \subset \Gamma\times\Gamma$, since Γ contains all the balls $B_\delta(x)$ for $x \in \partial\Omega$.

For all $1 \leq p < \infty$ and E a Lebesgue measurable set, $E \subset (\Omega\times\Gamma)^2$ consider the space of weighted L^p_μ functions:

$$L^p_\mu(E) := \{f : (\Omega\cup\Gamma)^2 \to \mathbb{R} | f \text{ measurable }, \int_E |f(x, x')|^p \mu(x, x')\, dx' dx < \infty\}.$$

Notice that these functions are defined over the entire domain $\Omega\cup\Gamma$, but they are L^p_μ integrable only on the set E. Due to the definition of N_μ we have that $L^p_\mu(N_\mu) = L^p_\mu[(\Omega\cup\Gamma)^2]$. We have that $L^p_\mu(E)$ is a vector space (in fact, Banach space – see Remark 3.3) with respect to the norm

$$\|f\|_{L^p_\mu(E)} := \left(\int_E |f(x, x')|^p \mu(x, x')\, dx' dx\right)^{1/p}.$$

Remark 3.2. Note that the equivalence classes of functions on $L^p_\mu(E)$ distinguish only among values for functions inside N_μ, since outside N_μ the weight $\mu = 0$. Thus

$$\|f\|_{L^p_\mu(E)} = 0 \text{ iff } f = 0 \text{ a.e. on } N_\mu \cap E.$$

Remark 3.3. Note that since μ is a nonnegative and measurable, the set function

$$\lambda(E) := \int_E \mu(x, x') dx dx'$$

is a measure on the Lebesgue measurable sets E subsets of $(\Omega\cup\Gamma)\times(\Omega\cup\Gamma)$ (according to Theorem 1.47 pg. 17 in [1]). (Observe that the measure λ may not be absolutely continuous with respect to the Lebesgue measure, since the kernel μ could be non-integrable). Thus $L^p_\mu(E)$ is the space of $p-$ integrable functions with respect to the new measure $\mu(x,x')dxdx'$, hence it is a Banach space.

For $u : \Omega \cup \Gamma \to \mathbb{R}$ denote

$$\hat{u}(x,x') := u(x) - u(x'). \tag{3.1}$$

Using this notation define the subspace of the functions with a zero "trace":

$$\mathcal{W} := \{w : \Omega \cup \Gamma \to \mathbb{R} | \hat{w} \in L^p_\mu[(\Omega \cup \Gamma)^2], \text{ and } w = 0 \text{ on } \Gamma\} \tag{3.2}$$

endowed with the norm

$$\|w\|_{\mathcal{W}} = \|\hat{w}\|_{L^p_\mu[(\Omega\cup\Gamma)^2]} = \left(\int_{\Omega\cup\Gamma} \int_{\Omega\cup\Gamma} |w(x') - w(x)|^p \mu(x,x')\, dx' dx \right)^{1/p}. \tag{3.3}$$

Remark 3.4. From Remark 3.2 we have that $\|u\|_{\mathcal{W}} = 0$ implies that $u =$ constant on N_μ, and due to the boundary conditions we have that $u = 0$ on N_μ. Observe that u may not be constant outside N_μ; however, the values of u outside N_μ are irrelevant due to the fact that u is integrated against μ, and $\mu = 0$ on the complement of N_μ.

A version of the following nonlocal Poincaré-type inequality was proved in [2]; this inequality will aid in proving coercivity for the functional $\mathcal{F}[u]$. Other nonlocal versions of Poincaré's inequality may be found in [3] and [21].

Lemma 3.5. *(Nonlocal Poincaré's Inequality.) Let $g \in L^p(\Gamma)$ with $p > 1$, $n \geq 1$. If $u \in g + \mathcal{W}$, and $\mathcal{G}$ is as defined in (2.1), then there exist $\lambda_{Pncr} = \lambda_{Pncr}(\Omega, \Gamma, \delta, n) > 0$ and $C_g > 0$ (unless $g \equiv 0$) such that the following inequality holds:*

$$\lambda_{Pncr} \|u\|_{L^p(\Omega)} \leq \|\mathcal{G}(u)\|_{L^p[(\Omega\cup\Gamma)^2]} + C_g, \tag{3.4}$$

which can also be written as

$$\lambda_{Pncr} \|u\|_{L^p(\Omega)} \leq \|\hat{u}\|_{L^p_\mu[(\Omega\cup\Gamma)^2]} + C_g. \tag{3.5}$$

Moreover, if $g \equiv 0$, then $C_g = 0$.

Remark 3.6. The above Lemma has as a consequence the fact that for *every* $p > 1$ one has

$$u \in \mathcal{W} \Rightarrow u \in L^p(\Omega).$$

Remark 3.7. In the classical case Poincaré's inequality yields higher integrability than L^p for every $u \in H^1_0(\Omega)$:

$$\|u\|_q \leq C \|\nabla u\|_p, \quad 1 \leq q \leq \frac{pn}{n-p}.$$

For the nonlocal theory, however, we can only obtain L^p integrability for u as a consequence of the L^p bound for $\mathcal{G}(u)$. This is the case also with the Poincaré's type inequality obtained in [2].

Proof. Proposition 4.1 from [2] yields the following inequality:

$$\lambda \delta^n \int_\Omega |u(x)|^p dx \leq \int_\Omega \int_{\Omega\cup\Gamma} \chi_\delta(x,x') |u(x') - u(x)|^p dx' dx + \delta^n \int_\Gamma |g(x')|^p dx',$$

where $\lambda = \lambda(n,p,\Omega) > 0$ and $n \geq 1$.

Applying assumption (2.8) we have the following:

$$\begin{aligned}\lambda\delta^n\int_\Omega |u(x)|^p dx &\le \int_\Omega\int_{\Omega\cup\Gamma}\chi_\delta(x,x')|u(x')-u(x)|^p dx'dx + \delta^m\int_\Gamma |g(x')|^p dx' \\ &\le \int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}\chi_\delta(x,x')|u(x')-u(x)|^p dx'dx + C_g \\ &\le \frac{1}{C_0}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}\mu(x,x')|u(x')-u(x)|^p dx'dx + C_g \\ &= \frac{1}{C_0}\|\mathcal{G}(u)\|^p_{L^p(\Omega\cup\Gamma\times\Omega\cup\Gamma)} + C_g\end{aligned}$$

Hence we have $\lambda_{Pncr}\|u\|_{L^p(\Omega)} \le \|\mathcal{G}(u)\|_{L^p(\Omega\cup\Gamma\times\Omega\cup\Gamma)} + C_g$, where

$$\lambda_{Pncr} = \lambda\delta^n C_0 > 0, \quad C_g = \delta^n\int_\Gamma |g(x')|^p dx'. \tag{3.6}$$

□

3.1. **Nonlocal Dirichlet's Priniciple.** Let us demonstrate that a solution (in the sense of distributions) of the boundary value problem (1.1) can be characterized as the minimizer of an appropriate functional.

Let $b \in L^q(\Omega)$ and $g \in L^p(\Gamma)$ where $\frac{1}{p}+\frac{1}{q}=1$, and $1 < p,q < \infty$. Define the functional:

$$\begin{aligned}\mathcal{F}[u] &:= \frac{1}{p}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|u(x')-u(x)|^p\mu(x,x')dx'dx + \int_\Omega b(x)u(x)dx \\ &= \frac{1}{p}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)(x,x')|^p dx'dx + \int_\Omega b(x)u(x)dx,\end{aligned} \tag{3.7}$$

for u in the class of admissible functions

$$\mathcal{A} = \{u : \Omega\cup\Gamma \to \mathbb{R} | u = g + u_0, \text{ for some } u_0 \in \mathcal{W}\}, \tag{3.8}$$

which by convention will be denoted by $g + \mathcal{W}$. This ensures that the admissible functions have their "trace" equal to g on Γ.

Theorem 3.8. *(Dirichlet's Principle)*

(i) *Assume u solves the nonlocal peridynamics problem (1.1). Then*

$$\mathcal{F}[u] \le \mathcal{F}[v] \tag{3.9}$$

for every $v \in \mathcal{A}$.

(ii) *Conversely, if $u \in \mathcal{A}$ satisfies (3.9) for every $v \in \mathcal{A}$, then u solves the nonlocal peridynamics problem (1.1).*

Proof. (i) Choose $w \in \mathcal{A}$. For u a solution to (1.1) we see that $u - w = 0$ over Γ. Applying Lemma 2.2 we obtain

$$\begin{aligned}0 &= \int_\Omega(\mathcal{L}_p u - b)(u-w)dx \\ &= -\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)|^{p-2}\mathcal{G}(u)\cdot\mathcal{G}(u-w)dx'dx - \int_\Omega b(u-w)dx\end{aligned}$$

By expanding the integrands and applying Young's inequality we obtain

$$\begin{aligned}
&\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)|^p dx'dx+\int_\Omega bu\,dx\\
=\ &\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)|^{p-2}\mathcal{G}(w)\cdot\mathcal{G}(u)dx'dx+\int_\Omega bw\,dx\\
\leq\ &\frac{p-1}{p}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)|^p dx'dx+\frac{1}{p}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(w)|^p dx'dx+\int_\Omega bw\,dx.
\end{aligned}$$

Applying (2.1) and rearranging we see $\mathcal{F}[u]\leq\mathcal{F}[w]$, for every $w\in\mathcal{A}$.

(ii) Conversely, suppose (3.9) holds. Fix $v\in\mathcal{W}$ and write $i(\tau):=\mathcal{F}[u+\tau v]$, where $\tau\in\mathbb{R}$. We have

$$i(\tau)=\frac{1}{p}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)+\tau\mathcal{G}(v)|^p\,dx'dx+\int_\Omega b(u+\tau v)dx$$

Since $u+\tau v\in\mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero. Thus $i'(0)=0$, $\left('=\frac{d}{d\tau}\right)$, provided the derivative exists.
Applying Lemma 2.2 we have

$$0=i'(0)=\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)|^{p-2}\mathcal{G}(u)\cdot\mathcal{G}(v)dx'dx+\int_\Omega bv\,dx=-\int_\Omega(\mathcal{L}_p u-b)v\,dx$$

Since v was an arbitrarily assigned function in $\mathcal{W}$, which is dense in every L^p when $\mu(x,x')\sim|x-x'|^{-\beta}$ with $\beta<p+n$ (since it contains the space of C^∞ functions with compact support on Ω), then $\mathcal{L}_p(u)(x)=b(x)$ in Ω in the sense of distributions. Note that if $\beta\geq p+n$, then the equation is trivially satisfied since the function $u\in\mathcal{W}$ must be identically zero.

□

Remark 3.9. Note that for the homoegeneous case, when the forcing $b=0$, Dirichlet's Principle holds for operators $\mathcal{L}_p$ with *very general* kernels μ that are only imposed to be symmetric and nonnegative, since our bound (2.8) on μ is not required for the proof. In the peridynamic setting this means that one may assume that the horizon of interaction $\mathcal{H}_x$ may have an arbitrary radius δ that varies with x or may have a different geometry (other than radial).

3.2. **Existence of minimizers.** In this section we will show that the minimizers to the functional $\mathcal{F}[u]$ exist, and they are unique. This will give the wellposedness of the system (1.1) by the Dirichlet's Principle.

We will state the following lemma; its proof is a straightforward verification which we will omit.

Lemma 3.10. *The mappings $u\to\mathcal{G}[u]$ and $u\to\mathcal{F}[u]$ are strictly convex.*

The main result of this paper is contained in the following theorem.

Theorem 3.11. *Let $b\in L^q(\Omega)$ and $g\in L^p(\Gamma)$, where $1<p,q<\infty$ are conjugate. If $\mathcal{F}[u]$ is defined as in (3.7) by*

$$\mathcal{F}[u]=\frac{1}{p}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma}|\mathcal{G}(u)(x,x')|^p dx'dx+\int_\Omega b(x)u(x)dx,$$

then

$$inf\{\mathcal{F}[u]:u\in\mathcal{A}\}\tag{3.10}$$

attains its minimum on a minimizer $\bar{u}$; furthermore, this minimizer is unique.

Proof. **Existence.** Note that $\mathcal{F}[u] \geq 0$ for every $u \in \mathcal{W}$. Write

$$inf\{\mathcal{F}[u] : u \in \mathcal{A}\} = m.$$

Observe that $m < \infty$ since the function $\mathcal{F}$ has a finite value for the function which is zero in Ω and equals g on Γ.
Let $\{u_\nu\}$ be a minimizing sequence so that $\mathcal{F}[u_\nu] \to m$. Thus there exists $M > 0$ such that $\mathcal{F}[u_\nu] < M$. We will show that the sequence u_ν is bounded in L^p. We apply Young's inequality with constants depending on ε (to be chosen later) so we have

$$\begin{aligned} M &> \frac{1}{2}\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} |\mathcal{G}(u_\nu)|^p dx' dx + \int_\Omega b(x) u_\nu(x) dx \\ &\geq \tfrac{1}{2}\|\mathcal{G}(u_\nu)\|^p_{L^p[(\Omega\cup\Gamma)^2]} - C(\varepsilon)\|b\|^q_{L^q(\Omega)} - \varepsilon\|u_\nu\|^p_{L^p(\Omega)}, \end{aligned}$$

for $p > 1$ finite and $1/p + 1/q = 1$. Applying Lemma 3.5 we obtain

$$M \geq \|\hat{u}_\nu\|^p_{L^p_\mu[(\Omega\cup\Gamma)^2]} - C(\varepsilon)\|b\|^q_{L^q(\Omega)} - \varepsilon A\|\hat{u}_\nu\|^p_{L^p_\mu[(\Omega\cup\Gamma)^2]} \tag{3.11}$$

where $A > 0$ depends on the Poincaré's constant λ_{Pncr}. By choosing ε sufficiently small we can find $\gamma > 0$ such that

$$\|\hat{u}_\nu\|_{L^p_\mu[(\Omega\cup\Gamma)^2]} \leq \gamma. \tag{3.12}$$

Therefore, we may extract a subsequence (unrelabeled) $\{\hat{u}_\nu\}$ such that

$$\hat{u}_\nu \to \bar{u} \text{ in } L^p_\mu[(\Omega\cup\Gamma)^2], \text{ for some } \bar{u} \in L^p_\mu(\Omega\cup\Gamma).$$

By Mazur's Lemma, there exists $\bar{u}_\nu = \sum_{i=j}^{N_j} \lambda_i^{(j)} u_i$, where for all $j, \sum \lambda_i^{(j)} = 1$ and $\lambda_i^{(j)} \geq 0$, such that $\bar{u}_\nu \to \bar{u}$ in $L^p_\mu[(\Omega\cup\Gamma)^2]$. Then employing Lemma 3.10 we have

$$\begin{aligned} \mathcal{F}[\bar{u}_\nu] &= \mathcal{F}[\textstyle\sum_{i=j}^{N_j} \lambda_i^{(j)} \hat{u}_i] \\ &\leq \textstyle\sum_{i=j}^{N_j} \lambda_i^{(j)} \mathcal{F}[\hat{u}_i] \\ &\leq (\textstyle\sum_{i=j}^{N_j} \lambda_i^{(j)})\mathcal{F}[\hat{u}_\nu] \\ &= \mathcal{F}[\hat{u}_\nu]. \end{aligned}$$

By passing to the limit in the above inequality, since $\hat{u}_\nu$ is a minimizing sequence, we have

$$\lim_{\nu\to\infty} \mathcal{F}[\bar{u}_\nu] \leq m.$$

By Fatou's lemma

$$\mathcal{F}[\bar{u}] \leq \lim_{\nu\to\infty} \mathcal{F}[\bar{u}_\nu],$$

hence, from the above inequalities we have that

$$\mathcal{F}[\bar{u}] \leq m.$$

Note also that $\bar{u}$ satisfies the boundary conditions, since it is the limit of the minimizing sequence $\hat{u}_\nu$ where $\hat{u}_\nu = g$ on Γ. This shows that $\bar{u}$ is a minimizer of (3.10).

Uniqueness. Assume that $\bar{u}, \bar{v} \in \mathcal{W}$ are minimizers of $\mathcal{F}$ with $\bar{u} \neq \bar{v}$. Let $\bar{w} = \dfrac{\bar{u}+\bar{v}}{2}$. Notice that $\bar{w} \in \mathcal{W}$. Now by Lemma 3.10, we have

$$m \leq \mathcal{F}[\bar{w}] \leq \frac{1}{2}\mathcal{F}[\bar{u}] + \frac{1}{2}\mathcal{F}[\bar{v}] = m.$$

Hence $\bar{w}$ is a minimizer of $\mathcal{F}$. This implies that

$$\frac{1}{2}\mathcal{F}[\bar{u}] + \frac{1}{2}\mathcal{F}[\bar{v}] - \mathcal{F}[\bar{w}] = 0.$$

Thus

$$\int_{\Omega\cup\Gamma}\int_{\Omega\cup\Gamma} \frac{|\mathcal{G}(\bar{u})|^p}{2} + \frac{|\mathcal{G}(\bar{v})|^p}{2} - |\mathcal{G}(\bar{w})|^p \, dx' dx = 0.$$

However, by Lemma 3.10, the integrand is strictly positive, a contradiction. Thus $\bar{u} = \bar{v}$ in Ω, the desired uniqueness.

□

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