Solutions to assignment 5

1. Consider the unforced van der Pol equation
   \[ u''(t) + (u^2 - 1)u' + u = 0. \]
   (a) Write the equation as a system of 2 ODEs;
   **Solution:** Let \( x_1 := u \) and \( x_2 := u' \) then the equation can be written as
   \[
   \begin{align*}
   x_1' &= x_2 \\
   x_2' &= -x_1 - x_1^2 x_2 + x_2
   \end{align*}
   \]
   or \( X' = F(X) \), where \( X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \) and \( F(X) = \begin{bmatrix} x_2 \\ -x_1 - (x_1^2 - 1)x_2 \end{bmatrix} \)
   (b) Find the equilibrium points and study their stability by linearization.
   **Solution:** In order to find equilibrium solutions we must solve \( F(X) = 0 \). This gives the only equilibrium point to be \( x_1 = x_2 = 0 \).
   To linearize the system around the critical point, we must first find the Jacobian matrix
   \[
   J_F(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 & -x_1^2 + 1 \end{bmatrix}
   \]
   At \((0,0)\) the matrix is \( \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \) which has eigenvalues \( \lambda_{1,2} = 1/2 \pm i\sqrt{3}/2 \). Since \( \text{Re} \lambda_i > 0 \) we have that the linearized system is unstable, hence the nonlinear system is unstable around the origin as well.

2. Determine the stability of the equilibrium solutions of the system
   \[
   \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 + 4 \\ x_1x_2 \end{bmatrix}
   \]
   **Solution:** The critical points are \((0,2)\) and \((1,0)\). The Jacobian matrix of the right hand side is given by
   \[
   J_F(x_1, x_2) = \begin{bmatrix} -4 & -2 \\ x_2 & x_1 \end{bmatrix}
   \]
   At \((0,2)\) the matrix becomes \( \begin{bmatrix} -4 & -2 \\ 2 & 1 \end{bmatrix} \), a matrix with eigenvalues \( \lambda_1 = \lambda_2 = -2 < 0 \), hence the equilibrium solution \((0,2)\) is asymptotically stable (i.e. all solutions nearby will approach it as time \( t \to \infty \)).
   At \((1,0)\) the Jacobian matrix is given by \( \begin{bmatrix} -4 & -2 \\ 0 & 1 \end{bmatrix} \) which has eigenvalues \( \lambda_1 = -4 < 0 \) and \( \lambda_2 = 1 > 0 \). This means that the linearized system has a saddle point at the origin, hence the nonlinear system is unstable around \((1,0)\) (it will also be a saddle critical point).

3. Find the equilibria of the nonlinear system below and study their stability
   \[
   \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} u(v - 1) \\ 4 - u^2 - v^2 \end{bmatrix}
   \]
   **Solution:** There are four critical points: \((0, \pm 2)\) and \((\pm \sqrt{3}, 1)\).
The Jacobian matrix is
\[
\begin{bmatrix}
v - 1 & u \\
-2u & -2v
\end{bmatrix}
\]

At \((0, 2)\) the Jacobian matrix is
\[
\begin{bmatrix}
1 & 0 \\
0 & -4
\end{bmatrix}
\]
which has eigenvalues \(\lambda_1 = -4 < 0\) and \(\lambda_2 = 1 > 0\), hence we have a saddle point which is unstable.

At \((0, -2)\) the Jacobian matrix is
\[
\begin{bmatrix}
-3 & 0 \\
0 & 4
\end{bmatrix}
\]
which has eigenvalues \(\lambda_1 = -3 < 0\) and \(\lambda_2 = 4 > 0\), again a saddle point (unstable).

At \((\sqrt{3}, 1)\) the Jacobian matrix is
\[
\begin{bmatrix}
0 & \sqrt{3} \\
2\sqrt{3} & -2
\end{bmatrix}
\]
which has eigenvalues \(\lambda_{1,2} = -1 \pm i\sqrt{5}\). Since \(\text{Re}\lambda < 0\) we have an asymptotically stable point (attractor/sink/focus).

At \((-\sqrt{3}, 1)\) the Jacobian matrix is
\[
\begin{bmatrix}
0 & -\sqrt{3} \\
2\sqrt{3} & -2
\end{bmatrix}
\]
which has again eigenvalues \(\lambda_{1,2} = -1 \pm i\sqrt{5}\); hence \((-\sqrt{3}, 1)\) is also an asymptotically stable point.

4. Consider the mass-spring nonlinear model with the equation
\[x''(t) + kx'(t) + g(x) = 0,\]
where \(g\) is continuous and it satisfies \(xg(x) > 0\) for \(x \neq 0\) and \(k > 0\) is the friction constant.

(a) Rewrite the equation as a nonlinear system of differential equations. Solution: The system is given by
\[
\begin{align*}
x' &= y \\
y' &= -ky - g(x).
\end{align*}
\]

(b) Show that the function
\[
V(x, y) := \frac{1}{2}y^2 + \int_0^x g(s)ds
\]
is a Lyapunov function. Is it a strict Lyapunov function?
Solution: Note: This problem was initially posted with a typo in the Lyapunov function, so everyone who attempted the problem will receive full credit for it. The solution was presented in full detail in class.
First note that we have \(g(-u) < 0 < g(u)\) for every \(u > 0\) hence, by continuity we must have that \(g(0) = 0\). This implies that the only critical point of the system is at the origin. The condition \(xg(x) > 0\) for every \(x \neq 0\) also implies that the function \(V\) given is positive definite. Thus, in order to establish stability for the origin we are left to check that \(V\) is decreasing along trajectories. We compute
\[
\frac{dV}{dt}(x(t), y(t)) = yy' + g(x)x' = y(-ky - g(x)) + g(x)y = -ky^2 \leq 0.
\]
Note, however, that the function is not strictly decreasing (we have that \(\frac{dV}{dt}(a, 0) = 0\) for any \(a \in \mathbb{R}\), hence we can not establish asymptotic stability (which holds for the system under the given assumption on \(g\)) with this Lyapunov function.

(c) Find an example that shows that the condition \(xg(x) > 0\) for \(x \neq 0\) is essential for stability.
Solution: If one considers \(g(x) = \sin x\) then the condition on \(g\) is not satisfied on the entire \(\mathbb{R}\). In fact, we have critical points \((k\pi, 0)\) for any \(k \in \mathbb{Z}\). We showed in class using linearization that the point \((\pi, 0)\) is unstable.

5. Use the Lyapunov function \(V(x) = \frac{1}{2}(x^2 + 3y^2)\) to show that the origin of the system
\[
\begin{align*}
x' &= -3y \\
y' &= x - \alpha(2y^2 - y)
\end{align*}
\]
is asymptotically stable for $\alpha < 0$.

**Solution:** Note first that the function $V$ is positive definite, hence it suffices to show that it is strictly decreasing to establish its asymptotic stability. Compute

$$\frac{dV}{dt} = xx' + 3yy' = x(-3y) + 3y(x - \alpha(2y^3 - y)) = -3\alpha y^4 + 3\alpha y^2 = -3\alpha y^2(2y^2 - 1).$$

Note that for $y$ close to the origin (we study the stability near the origin) we have that $y^2 < 1/2$ and since $\alpha < 0$ we obtain that $V$ is strictly decreasing (the only time the derivative is zero is at the origin, which is excluded, and at $y = 1/\sqrt{2}$ which is not close to the origin). Thus any solution that starts with $y_0 < 1/\sqrt{2}$ will be attracted by the origin which is an asymptotically stable point.

6. Consider the non-dimensionalized harmonic oscillator equation:

$$x''(t) + bx' + \sin x = 0.$$

Study the stability of the origin for all $b \geq 0$ by using the Lyapunov function given by the energy of the system, i.e. $V(x,y) = 1 - \cos x + \frac{y^2}{2}$, where $y = x'$.

**Solution:** The equation is written in the form of the system

$$\begin{cases} x' = y \\ y' = -by + \sin x. \end{cases}$$

One can show that $V$ is positive definite and decreasing along the trajectories as it follows from the computation:

$$\frac{dV}{dt} = (\sin x)x' + yy' = (\sin x)y + y(-by + \sin x) = -by^2$$

which is nonpositive for all $b \geq 0$ and all $y \in \mathbb{R}$. Hence the origin is stable.

As an alternative solution: apply directly the result of problem 4b) since $g(x) = \sin x$ satisfies $xg(x) > 0$ on the interval $(-\pi, \pi)$ and obtain that the origin is stable. The system is asymptotically stable for $b > 0$, but we would need a different Lyapunov function (or method – see the in-class solution using linearization) to show it.