1. (25 points) Let \( T(x, y, z) = \frac{3}{x^2 + y^2 + z^2} \), \((x, y, z) \neq (0, 0, 0)\).

(a) Find the equations of the tangent plane and normal line at the point \((-2, 1, 0)\) to the surface \( T(x, y, z) = \frac{3}{5} \).

**Solution:** We compute

\[
\frac{\partial T}{\partial x}(x, y, z) = -\frac{6x}{(x^2 + y^2 + z^2)^2}, \quad \frac{\partial T}{\partial y}(x, y, z) = -\frac{6y}{(x^2 + y^2 + z^2)^2}, \quad \frac{\partial T}{\partial z}(x, y, z) = -\frac{6z}{(x^2 + y^2 + z^2)^2}.
\]

At \((-2, 1, 0)\) we have

\[
\frac{\partial T}{\partial x}(-2, 1, 0) = \frac{12}{25}, \quad \frac{\partial T}{\partial y}(-2, 1, 0) = \frac{6}{25}, \quad \frac{\partial T}{\partial z}(-2, 1, 0) = 0.
\]

Therefore the normal line which has the direction of the gradient is given by

\[
(x, y, z) = (-2, 1, 0) + t\left(\frac{12}{25}, \frac{6}{25}, 0\right), \quad t \in \mathbb{R}.
\]

The equation of the tangent plane is \(\frac{12}{25}(x + 2) - \frac{6}{25}(y - 1) + 0(z - 0) = 0\) or \(2x - y + 5 = 0\).

(b) Find the rate of change of \( T \) at \((1, 2, 2)\) in the direction toward the point \((2, 1, 3)\).

**Solution:** To compute this directional derivative, we must first compute the unit vector \( u \) with the given direction. This is given by

\[
u = \frac{(2, 1, 3) - (1, 2, 2)}{||(2, 1, 3) - (1, 2, 2)||} = \frac{1}{\sqrt{3}}(1, -1, 1).
\]

Then

\[D_u T(1, 2, 2) = \nabla T(1, 2, 2) \cdot u = \left(-\frac{2}{27}, -\frac{4}{27}, -\frac{4}{27}\right) \cdot \frac{1}{\sqrt{3}}(1, -1, 1) = -\frac{10}{27\sqrt{3}}.
\]

(c) Show that at any point (except the origin) the direction of greatest increase is given by a vector that points towards the origin.

**Solution:** The direction of greatest increase at any point is given by the gradient, hence it is

\[
\left(-\frac{6x}{(x^2 + y^2 + z^2)^2}, -\frac{6y}{(x^2 + y^2 + z^2)^2}, -\frac{6z}{(x^2 + y^2 + z^2)^2}\right) = -\frac{6}{(x^2 + y^2 + z^2)^2}(x, y, z).
\]

Note that the vector \((x, y, z)\) is the position vector that points away from the origin, but since the scalar in front of the vector \(-\frac{6}{(x^2 + y^2 + z^2)^2}\) is always negative, the vector will point towards the origin.

(d) Let \( A := \{(x, y) \in \mathbb{R}^2 : T(x, y, 0) = \frac{1}{3}\} \). Use the implicit function theorem to find the points in \( A \) where \( y \) can be expressed as a function of \( x \), i.e. \( y = y(x) \); at those points compute \( dy \over dx \).

**Solution:** The equation \( T(x, y, 0) = \frac{1}{3} \) is equivalent to \( x^2 + y^2 = 9 \). Take \( f(x, y) = x^2 + y^2 \) and we see that its partial derivatives are continuous everywhere and \( f_y(x, y) = 2y \neq 0 \) everywhere where \( y \neq 0 \). This means that \( y \) can be written as a function of \( x \) everywhere, except for \( y = 0 \) (i.e. for the points \((\pm 3, 0)\)).
2. A rectangular box without a lid is to be made from 12 m\(^2\) of cardboard. Use the material from this course (gradient, second order derivative test) to find the maximum volume of such a box.

**Solution:** Let \(x\) and \(y\) be the length, respectively the width of the box, and \(z\) be the height. Then we have that the area of the box is \(A = xy + 2(x + y)z = 12\), from which it follows that

\[
z = \frac{12 - xy}{2(x + y)}
\]

The volume is \(V = xyz\) and by replacing \(z\) in terms of \(x, y\) we get

\[
V(x, y) = xy \cdot \frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)},
\]

hence \(V\) becomes a function of \(x, y\) only which can be maximized. To find the critical points of \(V\) we compute its gradient components and set them equal to zero:

\[
\frac{\partial V}{\partial x} = (12y - 2xy^2)(2(x + y)) - 2(12xy - x^2y^2) = y^2 \frac{12 - x^2 - 2xy}{2(x + y)^2}
\]

and

\[
\frac{\partial V}{\partial y} = (12x - 2x^2y)(2(x + y)) - 2(12xy - x^2y^2) = x^2 \frac{12 - y^2 - 2xy}{2(x + y)^2}
\]

We impose that the partial derivatives must be equal to zero, but reject that either \(x\) or \(y\) are zero, so we must have that \(12 - x^2 - 2xy = 12 - y^2 - 2xy\) from which it follows that \(x = y\) (since they must be positive). We obtain that \(x^2 = y^2 = 4\) hence \(x = y = 2\). from the equation for the area we obtain that \(z = 1\).

We obtained that \((2, 2)\) is a critical point, in order to check that it is a maximum for the function we use the second derivative test. We need to check that

\[
D_V(2, 2) := V_{xx}(2, 2)V_{yy}(2, 2) - [V_{xy}(2, 2)]^2
\]

is positive and \(V_{xx}(2, 2) < 0\).

We obtain that \(V_{xx}(2, 2) = -4, V_{x,y}(2, 2) = 2, V_{yy}(2, 2) = -4\) which gives indeed that \(D_V(2, 2) = 12 > 0\) with \(V_{xx}(2, 2) = -4 < 0\).
3. Use a double integral to find the volume of the tetrahedron bounded by the planes \( x + 2y + z = 2, \)
\( x = 2y, \) \( x = 0, \) and \( z = 0. \)

**Solution:** We will set up the integral as computing the volume under the surface \( z = 2 - x - 2y \) over the triangle \( 0 \leq x \leq 1, \ y \in [2x, 2 - 2x]. \) By looking at the figure we see that it is easiest to evaluate the integral in this order

\[
\int_{0}^{1} \int_{x/2}^{(2-x)/2} (2-x-2y) \, dy \, dx = \int_{0}^{1} (2y - xy - y^2) \big|_{y=x/2}^{y=(2-x)/2} \, dx = \int_{0}^{1} (1-2x+x^2) \, dx = (x-x^2 + \frac{1}{3}x^3) \big|_{0}^{1} = \frac{1}{3}
\]

**Alternative solution:** One could also compute the above volume by setting the integral up as

\[
\int_{0}^{1/2} \int_{0}^{2y} (2-x-2y) \, dx \, dy + \int_{1/2}^{1} \int_{0}^{2-2y} (2-x-2y) \, dx \, dy.
\]
4. Rewrite the integral
\[ \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx \]
as an iterated integral in the order \( dx \, dy \, dz \). Evaluate the integral (with any order of integration) for the constant function \( f(x, y, z) = 1 \).

**Solution:** We first evaluate the integral
\[
\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} 1 \, dz \, dy \, dx = \int_{-1}^{1} \int_{x^2}^{1} (1 - y) \, dy \, dx \\
= \int_{-1}^{1} (y - \frac{y^2}{2}) \big|_{y=x^2}^{y=1} \, dx = \int_{-1}^{1} \left( \frac{1}{2} - x^2 + \frac{x^4}{2} \right) \, dx = \left( \frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10} \right) \big|_{x=-1}^{x=1} = \frac{8}{15}
\]

To change the order of integration we must visualize or graph this three dimensional body. Since we have \( z \in [0, 1-y] \) we know that it is obtained from sectioning a body with the planes \( z = 1-y \) and \( z = 0 \). Since \( y \in [x^2, 1] \) we have that \( y \) is away from the parabolic cylinder \( y = x^2 \) (this cylinder is obtained by sliding up and down along the \( z \)-axis the parabola \( y = x^2 \)). Finally, the values for \( x \) are between \(-1\) and \( 1 \), which gives us the range of values for the parabolic cylinder.

Thus the integral can then be rewritten as
\[
\int_{0}^{1} \int_{0}^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy \, dz.
\]