Research Summary

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My primary research interest is in low-dimensional topology, particularly the topology of 3-dimensional manifolds.

There is a long tradition of using codimension-one objects to help understand the topology and geometry of 3-manifolds. The first and perhaps most successful such object was, and is, the embedded incompressible surface. In fact, nearly every open question in the theory of 3-manifolds has been settled for Haken manifolds, which are those that contain an incompressible surface. For example, Waldhausen proved in the 1960's that a Haken manifold M is determined up to homeomorphism by its fundamental group ('algebra determines topology'). More recently Thurston proved the geometrization theorem, which states that every Haken manifold can be canonically split open along tori so that the pieces are geometric: each admits a metric modeled on one of 8 model geometries ('topology determines geometry'). But since these results require an incompressible surface, such success has its limits; it is now generally accepted that most 3-manifolds, in some sense, do not contain incompressible surfaces.

Recently, more general codimension-one objects have been introduced to study 3-manifolds. Among them are π_1-injective immersed surfaces, taut foliations, and essential laminations. It is these last two that much of my research has focussed on.

Essential laminations are hybrid objects which lie between incompressible surfaces and taut foliations, and which generalize both. The idea is that they will provide tools for 3-manifold topology which are as powerful as incompressible surfaces have proven to be. But they have the further advantage of being far more common than incompressible surfaces. As an example, Delman and Roberts have shown that every non-trivial Dehn surgery along a non-torus alternating knot yields a manifold containing an essential lamination (or *laminar* manifold, for short). By contrast, Hatcher and Thurston showed that only finitely many Dehn surgeries, on each of a certain class of alternating knots known as 2-bridge knots, yield Haken manifolds.

The paragraphs that follow briefly describe four of the projects that I am currently working on.

Knot theory

Knot theory is one of the subjects in 3-manifold topology that has benefited the most from recent research into essential laminations. For example, since Gabai and Oertel proved that a laminar manifold has universal cover $\mathbf{R}^3$, Delman and Roberts' result immediately proves both the Property P Conjecture (non-trivial Dehn surgery never yields a simply-connected manifold) and the Cabling Conjecture (Dehn surgery on a non-cabled knot cannot yield a reducible manifold) for alternating knots. Finding essential laminations in knot exteriors, which survive (i.e., remain essential after) non-trivial Dehn surgery, thus provides one approach to solving such conjectures, for many classes of knots. Such laminations are called persistent. Recently I was able to show how one construction of an essential lamination in one knot complement, due to Oertel, could be adapted to find infinitely-many knots whose complements contain a persistent lamination. Since then, Roberts and I have found a more general construction, which gives many more examples of knots whose complements contain persistent laminations.

Exceptional fillings of hyperbolic 3-manifolds

One problem that has received much recent attention is the question of when the geometry (in the sense of Thurston's geometrization theorem) of a 3-manifold with boundary a 2-torus can degenerate (i.e., change) after Dehn filling - the act of gluing a solid torus to the boundary of the knot exterior. Thurston has shown that for a hyperbolic 3-manifold all but finitely many Dehn fillings remain hyperbolic. Much recent research has focused on determining when and how often other geometric structures arise when Dehn filling a hyperbolic manifold. Many general results are now known, with the single exception of when the Dehn-filled manifold might have $\mathbf{H^2} \times \mathbf{R}$ geometry and no essential tori; that is, when the manifold might be an exceptional Seifert-fibered space. Essential laminations provide a tool for answering this question, since there is a very good structure theory for essential laminations in Seifert-fibered spaces, worked out in my thesis. In particular, one can often tell, by looking at an essential lamination $\mathcal{L}$, that the manifold containing $\mathcal{L}$ is <u>not</u> Seifert-fibered. Together with constructions of essential laminations, due to Gabai and Mosher, this allowed me to show, for example, that at most 20 Dehn fillings on a hyperbolic manifold are Seifert-fibered. The previous best bound was 24, due to Bleiler and Hodgson; their result figured strongly in mine. Ying-Qing Wu and I have also used this structure theory to show that an exceptional Seifert-fibered space cannot occur after Dehn-filling on a 2-bridge knot exterior, unless the knot is a torus or twist knot. This leads to a complete classifiaction of Dehn surgeries on 2-bridge knots. This approach is quite general, so similar results should be obtainable wherever essential laminations can be constructed.

Algebra determines topology

One of the most important open questions in the theory of essential laminations is whether or not homotopy equivalent laminar manifolds are homeomorphic. In my previous work I showed that if a homotopy equivalence f:M→N, with N laminar, is sufficiently nice - specifically, if the pullback $f^{-1}(\mathcal{L})$, of an essential lamination $\mathcal{L}$ in N, is essential in M - then f is homotopic to a homeomorphism.

I am working to show that any homotopy equivalence is homotopic to one with an essential pullback; this has proved to be surprisingly difficult. It might be considered to be the most important unsolved problem in the theory of essential laminations. My most recent work on this problem has focussed instead on avoiding this step altogether, by exploring to what extent the requirement of essentiality is needed in the constructions of my previous work. Evidence suggests that it is not nearly as much as one might originally have thought.

A hyperbolic, non-laminar 3-manifold?

Laminar manifolds are far more plentiful than Haken manifolds. But not all 3-manifolds, even those with universal cover R^3, are laminar. The first examples were found in my thesis, where I showed that most exceptional Seifert-fibered spaces are not laminar. But to date no other examples are known; in particular, no hyperbolic example is known. The problem is that there is as yet no effective way to show that a 3-manifold doesn't contain an essential lamination (in contrast with the situation for incompressible surfaces). Using recent refinements in the structure theory for laminations in Seifert-fibered spaces, Naimi, Roberts, and I have established the intriguing result that the manifold obtained by 37/2 surgery on the (-2,3,7)-pretzel knot admits, essentially, exactly <u>one</u> essential lamination. This manifold is a graph manifold, however, and so is not hyperbolic. However, current sentiment favors the existence of a non-laminar, hyperbolic 3-manifold; some of the best prospects, in fact, seem to be among some of the other surgeries on the (-2,3,7) pretzel knot.

One possible approach to the non-existence question is to use normal forms. In previous work I have shown that for any triangulation τ of a laminar manifold M, there is an essential lamination which is in Haken normal form with respect to τ. This implies that, if there is an essential lamination in M, then there is one which is carried by one of a finite, constructible collection of normal branched surfaces. The last major hurdle in applying this result to find non-laminar 3-manifolds is the current lack of an effective procedure for determining when a branched surface carries a lamination; this is probably the other most important open problem in the theory of essential laminations. There are several sufficient conditions known, due to myself, Christy, and Naimi, as well as several well-known necessary conditions; the challenge is to bridge the gap between them.