PERSISTENTLY LAMINAR TANGLES

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ABSTRACT

We show how to build tangles T in a 3-ball with the property that any knot obtained by tangle sum with T has a persistent lamination in its exterior, and therefore has property P. The construction is based on an example of a persistent lamination in the exterior of the twist knot 6_1, due to Ulrich Oertel. We also show how the construction can be generalized to n-string tangles.

Keywords: Essential lamination, tangle, Dehn filling

0. Introduction

Essential laminations have proved very useful in understanding the topology of knots in the 3-sphere. Constructions of essential laminations in knot exteriors have allowed us to see that non-trivial surgery on non-torus alternating knots [DR], and on most algebraic knots [Wu] yield manifolds with universal cover $\mathbf{R}^3$, for example. This can be thought of as a (very) strong form of Property P for these knots. They can also provide a means of detecting the underlying geometric structure of the 3-manifolds obtained by surgery on a knot [Br1],[Br2],[BW].

In this paper we construct persistent laminations for knots, that is, essential laminations in the exterior of the knot, which remain essential after any non-trivial Dehn filling. Our starting point is a particularly simple example of such an essential lamination $\mathcal{L}$, found by Ulrich Oertel [Oe] in the complement of a twist knot (the knot 6_1 in Rolfsen's knot tables [Ro]), in connection with his work on laminations with a transverse affine structure. What we show here is that this lamination can be associated to a rather simple tangle T_0. By this we mean two things: (1) the lamination $\mathcal{L}$ lives in the complement of the tangle T_0 in the 3-ball B^3; (2) if we sum T_0 with any other tangle T to obtain a knot K in the 3-sphere S^3, then $\mathcal{L}$ is *persistent* for K. We call such a tangle *persistently laminar*. Being persisently laminar immediately implies, for example that every knot K obtained by tangle sum with T_0 has Property P. We also show that the construction of the lamination $\mathcal{L}$

can be generalized to provide many more examples of persistently laminar tangles.

1. The lamination

Oertel's construction of the lamination $\mathcal{L}$ begins with the branched surface B, depicted in Figure 1a, embedded in the complement of the 6_1 knot K_0. We have removed the knot K_0 in Figure 1b, to give a better view of the branched surface. This branched surface can be thought of as a once-punctured torus (i.e, a disk with a 1-handle attached), with its boundary glued to a curve running over the 1-handle, to create a single, embedded branch curve for B (as well as a second tube for the knot to run through); see Figure 1c.

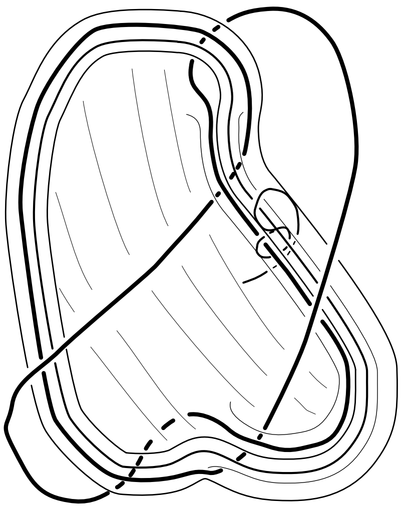

Figure 1a

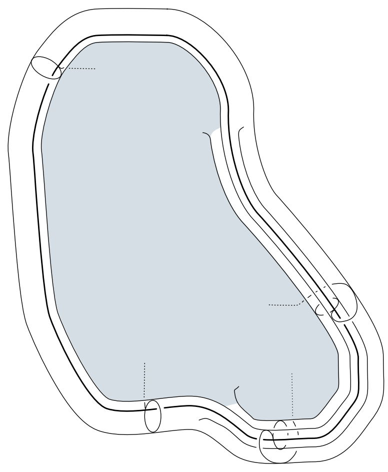

Figure 1b

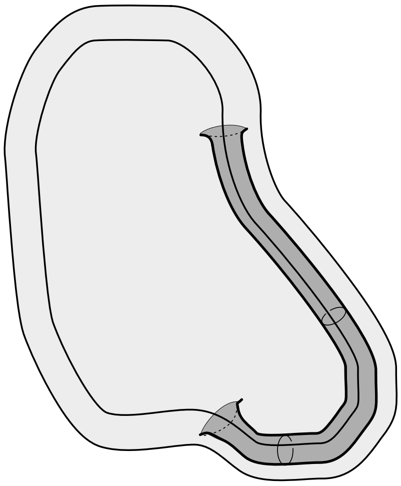

Figure 1c

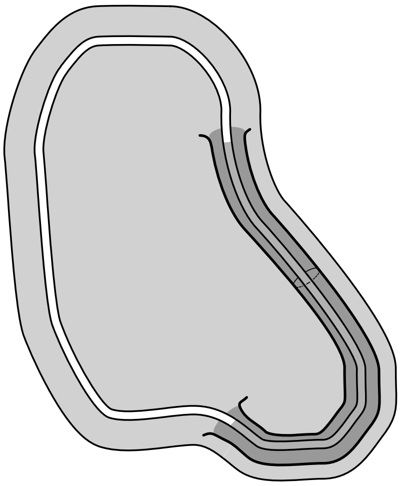

Figure 1d

Figure 2 provides a magnified picture of the process of gluing at the base of the 1-handle, to make it easier to see how the second tube is created.

Note that $N_0 = S^3\backslash\mathrm{int}N(B)$ is a genus-2 handlebody; this is most easily seen from Figures 1b and 2. By 'filling in' the two tubes that the knot K_0 runs through (one of which is created when we glue the boundary of the punctured torus to γ to create B), which we can think of as gluing two 2-disks D_1, D_2 to B, we can see that $N(B\cup D_1\cup D_2)$ is a 3-ball, i.e, $S^3\backslash\mathrm{int}N(B\cup D_1\cup D_2)$ is a 3-ball. N_0 is therefore a handlebody.

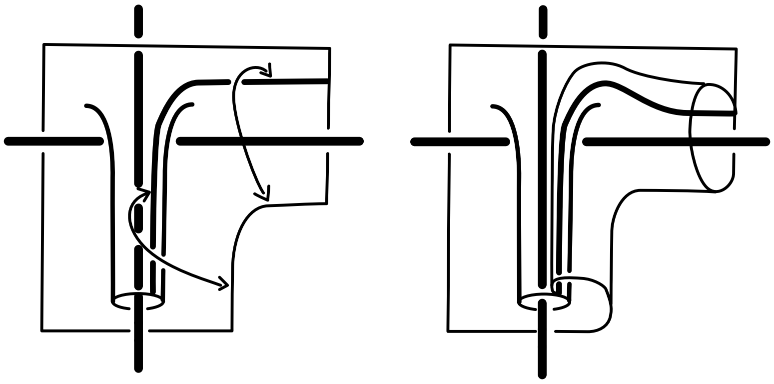

Figure 2

Next we show how to find a knot in the exterior of B so that B will be essential in the exterior of the knot. Our condition comes from the fact that our knot K_0 above meets each of the disks D_1, D_2 in one point.

THEOREM. *Let K be a knot in $N_0 = S^3\backslash N(B)$ meeting each of the disks D_1, D_2 in one point. Then B is an essential branched surface in $M = S^3\backslash int(N(K))$.*

Proof: For B to be essential we need to know 6 things:

(1) B carries a lamination $\mathcal{L}$ with full support.

This is immediate, since B has no triple points; the branch curve γ does not intersect itself. If we cut B open along γ (see Figure 1d), we get a surface with boundary, F. By taking a Cantors set's-worth of copies of F, embedded transverse to the fibers of $N(B)$, we can glue these surfaces together where the three copies of γ meet (since the concatenation of two Cantor sets is order isomorphic to a Cantor set) to create a lamination $\mathcal{L}$ carried with full support by B.

(2) B does not carry a 2-sphere, and B has no disks of contact.

This follows because B has only one sector, i.e., $B\backslash\gamma$ is connected. The sector is in fact a twice-punctured 2-disk F (it is, after all, a once-punctured torus cut open along a non-separating curve); see Figure 1d. Any surface carried by B would consist of finite number a of parallel copies of F, glued together where the three boundary components of F come together at γ. When these three sheets come together we get a consistency condition to determine if we can glue the boundary components together to get a closed surface (Figure 3). In this case the condition is $a + a = a$, implying $a = 0$. So no such surface exists. A disk of contact is similar; it is a 2-disk carried by B, whose boundary lives in the vertical boundary $\partial_v N(B)$ of B. This must again be built by gluing copies of F together, except this time, after gluing, a boundary component is left free. This gives the consistency condition $a + a + 1 = a$, implying $a = -1$, which is absurd. So there are also no disks of contact.

In point of fact, we have shown that B carries no closed surface, and has no compact surfaces of contact.

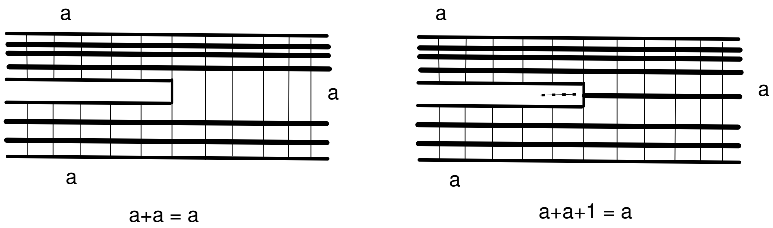

Figure 3

(3) B does not carry a compressible torus.

This follows from the above, since B does not carry a closed surface.

(4) $M_0 = M \backslash \mathrm{int}(N(B))$ does not have any monogons.

This is also immediate, because B is transversely orientable; there is a vector field (in M) everywhere transverse to the tangent planes of B. The arc in the boundary of a monogon which meets $\partial_h(N(B))$ is a transverse orientation-reversing loop.

(5) M_0 is irreducible.

Suppose S is a reducing sphere for M_0. Since N_0 is a handlebody, S bounds a 3-ball B^3 in N_0. Since this 3-ball cannot live in M_0, we must have $K \subseteq B^3$. But this implies that K is null-homotopic in N_0, hence homologically trivial. But K intersects D_1, for example, exactly once, and so its homology class has non-trivial intersection number with the class of D_1, so is non-trivial (Figure 4). So the reducing sphere cannot exist.

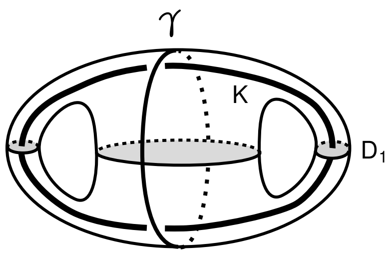

Figure 4

(6) The horizontal boundary $\partial_h N(B)$ of B is incompressible in M_0.

Again the idea is to use the fact that K pierces the disks D_i in one point each. The point is that there are very few compressing disks for $\partial_h N(B)$ in the handledbody

N_0 to begin with, and the fact that we are dealing with a knot means that all of them must intersect K.

Suppose D is a compressing disk for $\partial_h N(B)$ in M_0. Then in particular, it is a compressing disk for $\partial_h N(B)$ in N_0. The key to the argument is the fact that γ itself bounds a compressing disk, call it D'. because $\partial D \cap \gamma = \partial D \cap \partial D' = \emptyset$, we can, by a disk-swapping argument, make D and D' disjoint. Now $N_0 \backslash D'$ consists of two solid tori, each with a fat point (namely D') removed from their boundaries. D can therefore be thought of as a compressing disk for a solid torus which has a point removed from its boundary. It is therefore isotopic either to a meridian disk of the solid torus (and therefore, back in N_0, is isotopic to one of the disks D_1 or D_2), if ∂D is essential in the boundary of the solid torus, or is boundary parallel, i,e, is parallel to a disk in the boundary of the solid torus, with the removed point in its interior; see Figure 5a. Back in N_0 , this second disk is parallel to D'.

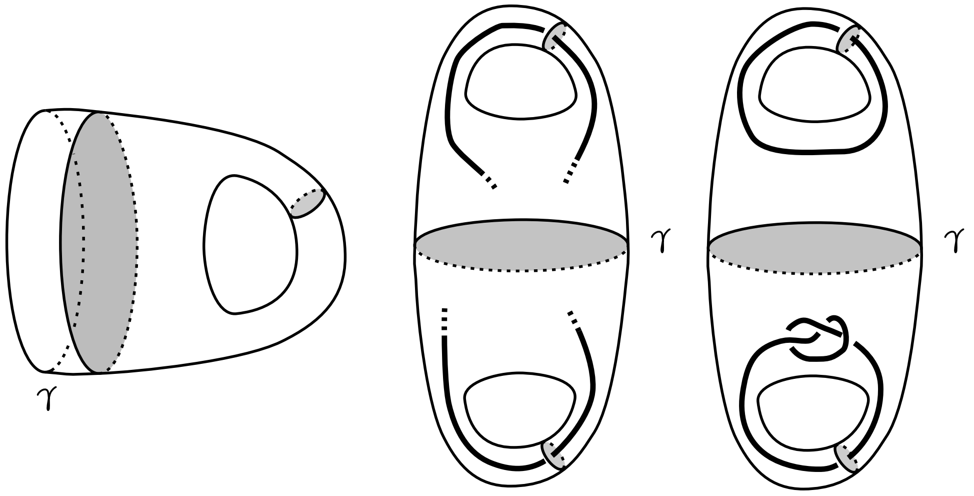

Figure 5

But both of these possibilities are absurd; in the first case D, which misses K, is isotopic, rel boundary (if we wish - the original isotopy moved ∂D around in ∂N_0), to a disk which hits K exactly once, contradicting the invariance of intersection number for homology classes (Figure 5b). In the second case D separates N_0, yet K, which is connected, has non-trivial intersection with each piece (Figure 5c). Therefore, no compressing disk for $\partial_h N(B)$ in M_0 exists.

Consequently, all properties of essentiality are satisfied, so B is an essential branched surface, and $\mathcal{L}$ is an esential lamination, in $S^3 \backslash \mathrm{int}(N(K))$. $\square$

2. The tangle T_0

$N(B \cup D_1 \cup D_2)$ is a 3-ball B_0, which the our original twist knot K_0, and each of the knots K, intersects in a pair of arcs, namely the two cores of the tubes that the disks cap off. In other words, each K meets this 3-ball in the same tangle T_0. But we have not yet identified this tangle. To see what it is, imagine stretching the two arcs in the 3-ball to fill the two tubes of $S^3\setminus\mathrm{int}(N(B))$; we arrive at a picture as in Figure 6a (we have also included the branch curve γ for reference). The key point here is that the arc which, in this picture, crosses <u>over</u> the other, comes out of the 3-ball in <u>back</u>. Pulling it around front adds extra half-twists to the tangle, so that we end up with the tangle in Figure 6c. It is the sum of two rational tangles, the $1/3$ tangle and the $-1/3$ tangle.

Our original knot K_0 is obtained from this tangle by tangle sum with a rational tangle; see Figure 6d.

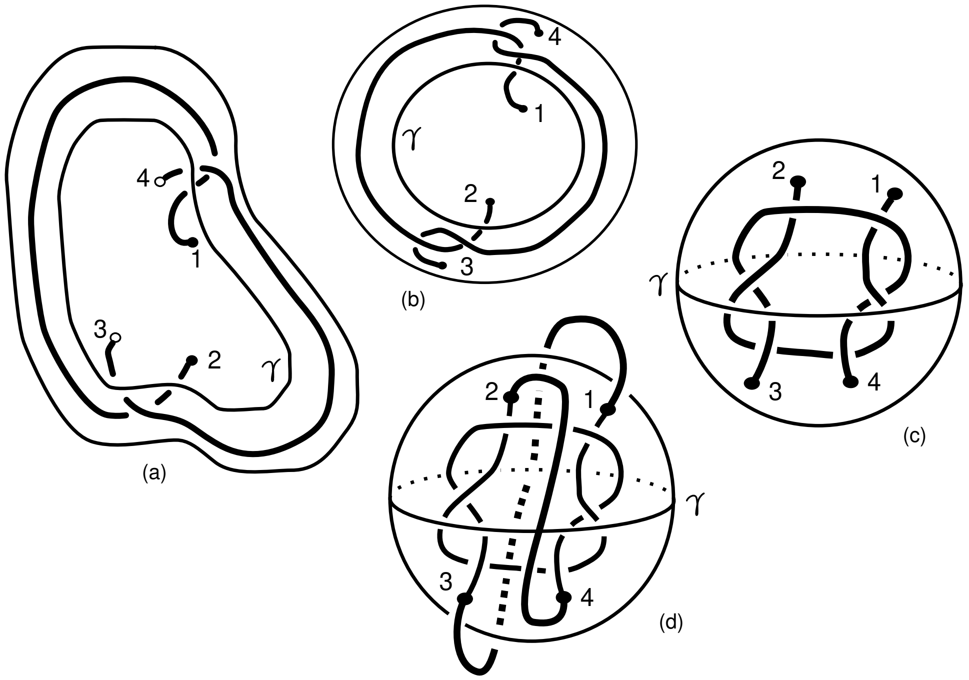

Figure 6

3. Persistence

The analysis of Section 1 above shows that the lamination $\mathcal{L}$ is essential in the complement of any knot K obtained by tangle sum of T_0 with any other tangle T. We now show:

PROPOSITION. *$\mathcal{L}$ is persistent for the knot K; it remains essential in any manifold obtained by non-trivial Dehn filling along K.*

Proof: For basic concepts on Dehn filling and Dehn surgery, the reader is referred to [Ro].

To show that $\mathcal{L}$ is essential in the manifold $K(r)$ obtained by r-Dehn-filling along K, for $r \neq 1/0$, we will verify the six properties (1)-(6) above, in this new setting. Once again, the first four of these properties require no extra proof, since $N(B)$ has not changed; only where it is embedded has. So we need only show that $K(r)\setminus\mathrm{int}N(B) = M_r$ is irreducible, and $\partial_h N(B)$ is incompressible in M_r.

We can think of M_r as the result of r-Dehn-filling on K in the genus-2 handlebody N_0. Both of our proofs will rely on the fact that $N_0\setminus\mathrm{int}(N(K))$ contains two embedded annuli $A_i = D_i\setminus\mathrm{int}(N(K))$, i=1,2, each with one component on $\partial_h N(B)$ and the other a meridional loop on $\partial N(K)$.

To show that M_r is irreducible, suppose it is not. Then there is a 2-sphere S in M_r which does not bound a ball in M_r. Choose such a sphere which intersects (transversely) the (image of the) knot K in the fewest number of points. It is then standard that $S\setminus\mathrm{int}(N(K)) = S'$ is an incompressible and ∂-incompressible planar surface in $N_0\setminus\mathrm{int}(N(K)) = M_0$. The curves $S'/\mathrm{ap}\partial N(K)$ are parallel curves of slope r.

Look at $S'\cap A_i$; it consists of circles and arcs. Trivial circles of intersection can be removed by isotopy, since S' is incompressible. The arcs of intersection cannot meet the boundary component of A_i coming from ∂N_0, since S misses ∂M_r. These arcs of intersection are therefore boundary parallel, and so can also be removed by isotopy, since S' is ∂-incompressible. After these isotopies, if $\partial S' \neq \emptyset$, $\partial S'\subseteq\partial N(K)$ misses a meridional loop, and hence consists of meridional loops. So $r = 1/0$, a contradiction.

Therefore $\partial S'$=$\emptyset$, i.i., $S' = S$. But since N_0 is irreducible, S bounds a 3-ball in N_0. This 3-ball must intersect, hence contain, K, since otherwise S bounds a 3-ball in M_r. But this implies that K is null-homologous in N_0, which is impossible since it intersects a compressing disk D_1 of ∂N_0 exactly once. So the reducing sphere cannot exist; M_r is irreducible.

To prove incompressibility of ∂_h, we again appeal to the two disjoint annuli A_i in M_0, joining meridional loops in $\partial N(K)$ to loops in $\partial_h N(B)$. The two loops in $\partial_h N(B)$ are obviously not homotopic to one another on $\partial_h N(B)$; they lie in different components. We have already seen above that $\partial_h N(B)$ is incompressible in $N_0\setminus\mathrm{int}(N(K))$. It then follows from Theorem 4 of [Me] that $\partial_h N(B)$ will remain incompressible in any manifold obtained by non-trivial (i.e., non-meridional) Dehn filling along K (in N_0).

Therefore, all of the properties of essentiality for B (and hence for $\mathcal{L}$) are satisfied in any manifold obtained by non-trivial Dehn filling along any of the knots K. So K is persistently laminar. $\square$

4. Generalizations

The existence of a persistent lamination in the complement of knots K obtained from T_0 duplicates previous work. If T is a rational tangle, then K is a Montesinos knot, and for such tangles Delman [De] has constructed persistent laminations for the resulting knots. On the other hand, if T is an non-split tangle, then Wu [Wu] has shown that K admits a persistent lamination. These two results have very powerful generalizations, as well. The intersection of the complements of these two classes of tangles is the collection of split, non-rational, tangles, and so the resulting knots are all connected sums with a square knot. One of the swallow-follow tori for each knot will then remain incompressible under all non-trivial Dehn fillings.

The technique for building the branched surface B that we have used here can, however, be easily extended to more than one tube; see Figure 7 for the case n=2. We can then string arcs through the tubes of B_n, to create a tangle of 2n arcs in a 3-ball. $S^3 \backslash \mathrm{int}(N(B_n)) = N_n$ is again a handlebody (it is a 3-ball with $2n$ 1-handles attached,) and the branch curve γ separates the 1-handles into two collections of n each (Figure 8). In this case however, we cannot always add arcs in the central 3-ball to create a knot in any way we choose, and still expect B_n to be essential; an arc running (parallel to ∂N_n) from arcs on the same side of γ , for example, has a (parallel) ∂-compressing disk around it (see Figure 8b).

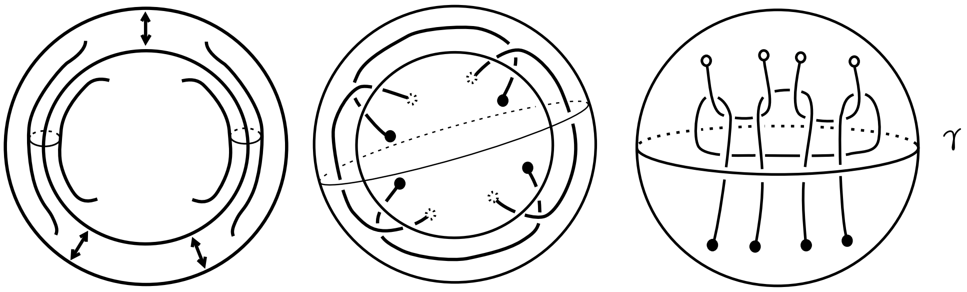

Figure 7

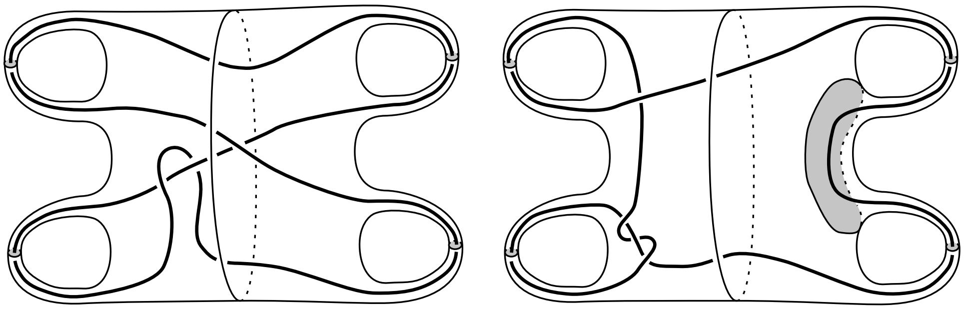

Figure 8

However, incompressibility of the horizontal boundary $\partial_h N(B_n)$ in N_n is the only obstruction to the essentiality of B_n, as well as remaining essential under any non-trivial Dehn surgery. The first four conditions on essentiality follow the exact same line as in our original case. We also still have the meridional annuli which allow us to verify that irreducibility and incompressibility of $\partial_h N(B_n)$ will be inherited under any non-trivial Dehn filling. In some sense, it turns out, the phenomenon described above is also the only way to prevent incompressibility, as well.

We can push compressing disks off of the tubes, by pushing them off of the meridional annuli; see Figure 9. Trivial circles of intersection with A_i can be removed by isotopy, since N_n (and therefore $N_n \backslash K$) is irreducible, and then we may surger along arcs of intersection to create two disks, at least one of which has boundary non-trivial in $\partial_h N(B)$, giving us a new compressing disk with fewer intersections with A_i. Finally, we cannot have any circles of intersection which are essential in A_i, since surgering along the innermost one (using the disk in D_i that it bounds, which meets K once) would produce a disk and a 2-sphere (in S^3) each intersecting K exactly once. But a sphere in S^3 cannot meet a knot only once.

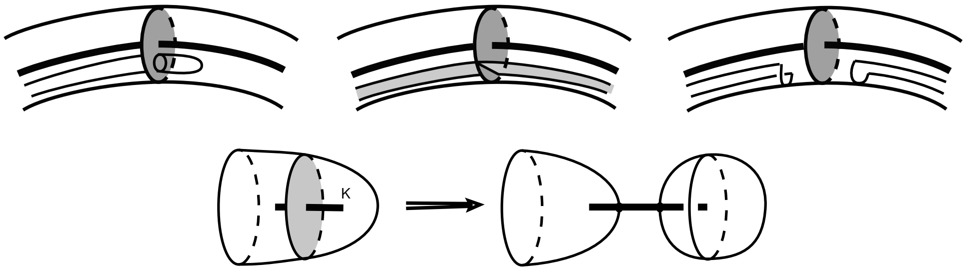

Figure 9

Our compressing disk D then lies in the central 3-ball piece B^3 of $S^3 \backslash \mathrm{int}(N(B_n))$. Again, we can assume (by disk-swapping) that D misses the obvious compressing disk D' that the branch curve γ bounds. D then splits B^3 into two 3-balls B_1^3 and B_2^3; one of them, B_1^3 say, misses D'. We must then have $K \cap B_1^3 \neq \emptyset$; otherwise, D can be isotoped, rel boundary, into $\partial_h N(B_n)$, since ∂B_1^3 misses K, hence misses the subdisks of ∂B^3 which the 1-handles of N_n are attached to (see Figure 10).

Therefore $K \cap B_1^3$ consists of some non-zero number of components of the $2n$-strand tangle $K \cap B^3$. These arcs are disjoint from B_2^3, and so are disjoint from D', and so each joins endpoints of core arcs of 1-handles which are on the same side (i.e., the D-side) of γ.

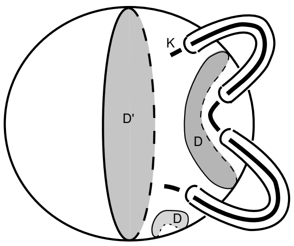

Figure 10

Therefore, one way to ensure that the branched surface B_n is essential in the complement of K, and remains esential under all non-trivial Dehn fillings, is to insist that all of the arcs we use to build K travel from one side of γ to the other, as in Figure 8a. This is still a vast number of knots, all of which admit a persistent lamination. Even more, we can allow ourselves to connect the ends of this $2n$-strand tangle as above to create links, as well. Since each component of the link must visit both sides of γ, each component comes equipped with two of the meridional annuli A_i, whose boundary components are on different components of $\partial_h N(B)$. Therefore, non-trivial Dehn filling on each component of the link (what is sometimes called a *complete* Dehn filling) yields a manifold in which our lamination remains essential. Note that, with our original tangle T_0, the condition that the arcs of the tangle T travel from one side of γ to the other is precisely the condition that the resulting link is in fact a knot. So this new condition is a natural extension.

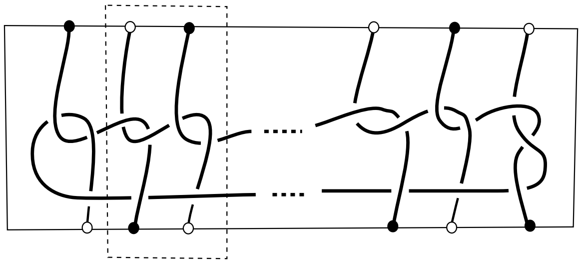

Figure 11

It is not hard to see that the tangle of Figure 7 (and its generalizations with more tubes) can be isotoped to an alternating diagram. Adding more arcs to the tangle (i.e., adding more tubes to B_n) simply amounts to grafting on an additional fundamental piece to the tangle, shown in the dotted rectangle in Figure 11. The condition above then amounts to requiring that the complementary $2n$-strand tangle join black-dotted ends to white-dotted ends. Joining together all but two pairs of ends produces an ordinary tangle. Note that each strand of the tangle must have been built from an odd number of our original strands, in order for its ends to lie on the same side of the compressing disk D. Since our lamination remains essential and persistent no matter how this tangle is completed to a knot, these tangles are persistently laminar.

5. Still more generalizations

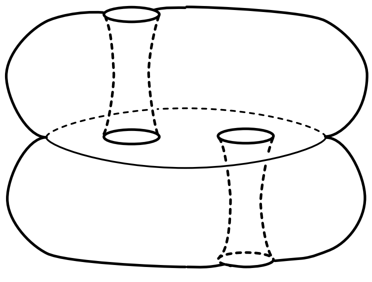

Figure 12a

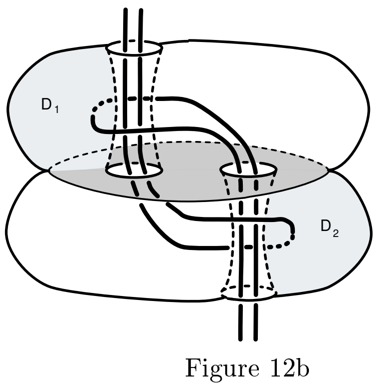

Figure 12b

Ramin Naimi has pointed out that the branched surface B which we began with can be drawn in a different (and ultimately more useful) way; see Figure 12. In this form it is easy to see all of the components of the construction which we have exploited; the compressing disk bounded by the branch curve γ, the compressing disks for the two 1-handles, and the tangle T_0 built from the core arcs of the 1-handles.

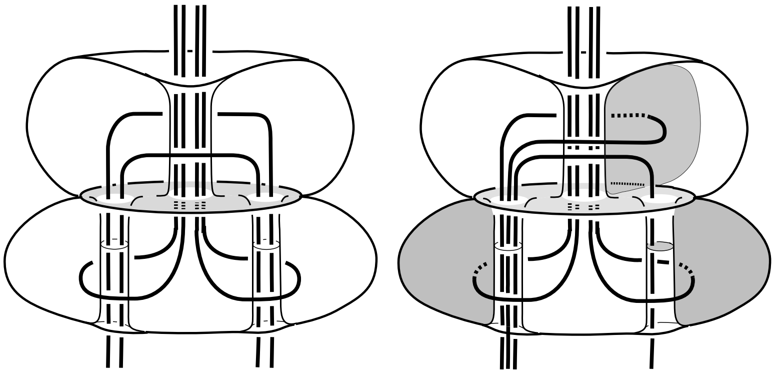

Figure 13

Our more general branched surfaces B_n have similarly simple pictures; see Figure 13a. In this form, however, it is also easy to see that there are <u>different</u> choices of how to write $S^3\setminus\text{int}B_n{=}N_n$ as a 3-ball B^3 (containing the branch curve γ) with 1-handles attached, by choosing <u>different</u> compressing disks for $\partial N_n\setminus\gamma$; see Figure 13b. We can take the core arcs of these compressing disks, and think of them as a $2n$-strand tangle in the complementary 3-ball $S^3\setminus B^3$. Structurally, these tangles have the exact same properties which we used in Section 4 to show that B_n is essential in the complement of any knot or link obtained by gluing on a tangle in B^3 all of whose strands cross the disk D bounded by γ. Therefore, we can obtain new examples of persistently laminar tangles by choosing sets of compressing disks for the two genus-n handlebodies of $N_n|D$, and taking the core arcs of the disks. We give a further example in Figure 14.

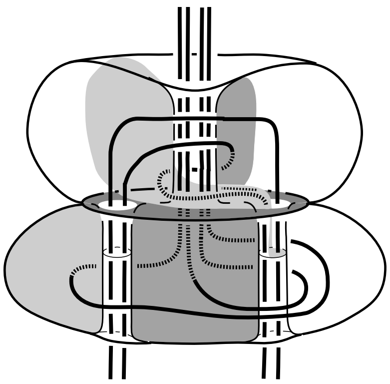

Figure 14

We can list of the properties of our branched surfaces B_n which we have used in our proofs; this gives us a recipe for finding persistently laminar tangles. We needed a transversely orientable branched surface B in S^3 having one branched curve γ, with no triple points, so that $B\backslash\gamma$ is connected. We also require that γ bounds a disk D which splits $S^3\backslash\text{int}N(B)$ into two genus-n handlebodies (note that the two handlebodies must have the same genera); in particular, $S^3\backslash\text{int}N(B)$ is a handlebody. Choosing compressing disks for each handlebody and taking their core arcs gives us a $2n$-strand tangle which we can string together as above to create persistently laminar tangles. In Figure 15 we provide an example, using this recipe.

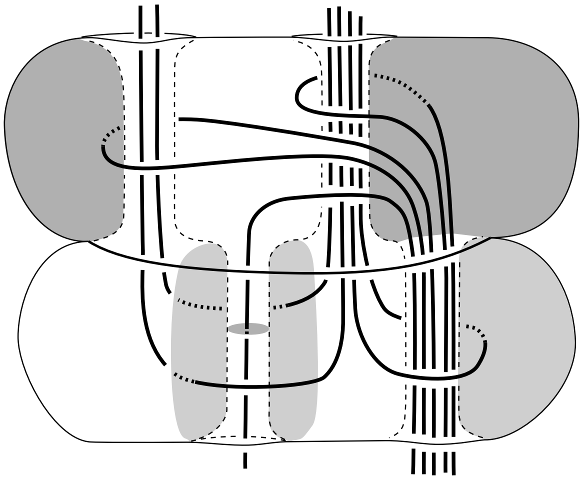

Figure 15

6. Concluding remarks

The tangles we have described here come equipped with a lamination in their tangle space (the 3-ball with the arcs of the tangle removed), which remains essential after non-trivial surgery on any knot constructed from the tangle. This is what we have called persistently laminar. One could weaken this definition, without losing its essential strength, by requiring instead that for every knot obtained from the tangle, there is a lamination which remains essential under non-trival Dehn filling. The work of Delman [De] and Wu [Wu] then demonstrate that many other tangles are persistently laminar in this sense; for example, the sum of two rational tangles whose associated rational numbers have denomentators at least 3 and have different signs (such as, for example, our tangle T_0) [De], or the sum of two atoroidal tangles [Wu]. Most algebraic tangles (see [Wu]) are also persistently laminar, in this weaker sense. The technique of the previous paragraph can easily provide examples of persistently laminar (alternating) tangles which cannot be decomposed (non-trivially) as the sum of two tangles, however, making them disjoint from these collections of tangles.

There are, of course, many tangles which are not persistently laminar; any tangle which can be summed to give a knot admitting a finite or reducible surgery, for example, cannot be persistently laminar, because the surgery manifold is not laminar. So, for example, no rational tangle is persistently laminar; each can be summed with another rational tangle to produce a (2,q)-torus knot. Other, more sporadic, examples can easily be given.

In this paper we have worked, in some sense, backwards, by building a lamination and then finding the tangle space which it should live in. A far more difficult (and so correspondingly rewarding) approach is to try to determine if a given tangle is persistently laminar, in either sense. Wu, for example, suggests the tangle of Figure 16 as an example; it is, in some sense, the smallest non-algebraic tangle. We do not know whether or not it is persistently laminar. No knot obtained from it by tangle sum with another tangle is known to fail to be persistently laminar.

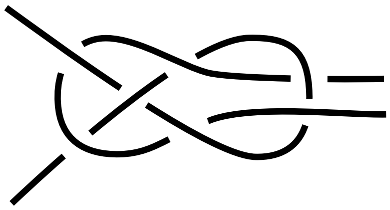

Figure 16

The laminations we have worked with are also in some sense the 'simplest' laminations one could build; their branched surfacees have a single branch curve with no triple points. It is remarkable how many knots these very simple laminations are persistent for; are there other constructions which are similarly powerful?

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