Essential Laminations in Non-Haken 3-manifolds

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In this paper we show that an essential lamination $\mathcal{L}$ in a non-Haken 3-manifold M is 'tightly wrapped', in the sense that any two leaves of $\mathcal{L}$ have intersecting closures; $\mathcal{L}$ therefore contains a unique minimal sublamination. We also show that these properties are inherited by any lift of $\mathcal{L}$ to a finite cover of M.

Essential laminations are natural generalizations of incompressible surfaces in a 3-manifold, and give topologists an object which can be found in many more 3-manifolds than an incompressible surface can (see, e.g., [De] or [Na]). Yet they retain enough properties in common with incompressible surfaces so that they can be used to prove some of the powerful results about 3-manifolds that incompressible surfaces have been used to prove (see, e.g., [G-O] or [G-K]).

This note was motivated by asking the question 'What <u>more</u> can be said about an essential lamination if we know that the 3-manifold M containing it is non-Haken, i.e., M does <u>not</u> contain a (2-sided) incompressible surface?' What we find is the following 'structure theorem':

Theorem 1: Let $\mathcal{L}$ be an essential lamination in a compact, non-Haken 3-manifold. Then $\mathcal{L}$ contains a unique non-empty minimal sublamination $\mathcal{L}_0$, i.e., a lamination $\mathcal{L}_0 \subseteq \mathcal{L}$

* Supported in part by NSF grant #DMS-8610730

with the property that $\mathcal{L}_0 \subseteq \overline{\mathrm{L}}$ for every leaf L of $\mathcal{L}$. $\mathcal{L}_0$ can therefore be defined (a posteriori) as $\mathcal{L}_0 = \bigcap_{\mathrm{L} \subseteq \mathcal{L}} \overline{\mathrm{L}} \neq \emptyset$.

Since a codimension-1 foliation without Reeb components is an example of an essential lamination, we have as an immediate consequence:

Corollary 2: A codimension-1 foliation $\mathcal{F}$ without Reeb components in a compact non-Haken 3-manifold M contains a unique minimal set.

This corollary is of independent interest; it is also an interesting example of a foliation-theoretic result which (seemingly) requires a lamination-theoretic proof. Previously (see [La] or [Co]) it was known that $\mathcal{F}$ has only a finite number of distinct minimal sets (that result does apply much more generally, however).

Thus (in a sense), in a non-Haken 3-manifold essential laminations must be 'tightly-wrapped'. This restriction on their structure adds a potentially useful new tool to the further study of essential laminations. In particular, since Haken manifolds are already fairly well-understood, this additional structure appears in exactly those 3-manifolds in which essential laminations will be the most useful: those which do not already contain an incompressible surface. It can therefore turn what might be perceived as a liability (the lack of an compact leaf, to which more classical techniques could apply) into an asset (extra information about the 'shape' of the lamination); see, e.g., [Br1]. This usefulness will have its limitations, however: in the second section we demonstrate its lack of application to finding Haken finite coverings of non-Haken 3-manifolds, by showing that the same tightness property is inherited by any lift of $\mathcal{L}$ to a finite cover of M.

1. Proof of Theorem 1:

The reader is referred to [G-O] for the basic concepts regarding essential laminations. We will assume throughout that the 3-manifold M in question is compact (hence closed, because M is non-Haken), connected, and orientable.

Proposition 3: If M is an irreducible 3-manifold with non-empty boundary, then either M contains a closed (2-sided) incompressible surface or M is a handlebody.

Proof: Choose a ∂-component F of M. If it is incompressible, we are done. Otherwise there is a compressing disk D for F, and we begin building a <u>compression</u> <u>body</u> (see [Bo]) for M along F by writing M= $M_0 \cup C_0$, where $C_0 = N(F \cup D)$ (i.e., it is N(F) with a 2-handle attached), and M_0 is the rest of M. ∂M_0 has boundary component $F_0 = \partial N(F \cup D) \backslash F$, and we again ask if this is incompressible in M_0. Continuing inductively, if at any stage F_i is incompressible in M_i, then it is incompressible in M, because F_i is also incompressible in C_i (turning it upside-down (i.e., turning its handlebody structure upside-down), C_i is basically $N(F_i)$ with a bunch of 1-handles attached, so F_i π_1-injects). Otherwise, we can decompose F down to a collection of sphere ∂-components of some M_n. It is easy to see that M_n is irreducible (inductively), and so M_n then consists of a union of 3-balls = 0-handles; Turning C_n upside-down then demonstrates that M is a union of 0-handles with 1-handles attached, i.e., is a handlebody. ∎

Lemma 4: Every lamination $\mathcal{L}$ contains a minimal sublamination, i.e., a non-empty lamination $\mathcal{L}_0 \subseteq \mathcal{L}$ which contains no proper (non-empty) sublamination.

Proof: Look at the collection of non-empty sublaminations of $\mathcal{L}$, ordered by inclusion. Because M is compact, any sequence $\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \ldots$ has a lower bound $\bigcap_i \mathcal{L}_i \neq \emptyset$ (any finite intersection is non-empty, so the full intersection is; it is a lamination because any leaf of $\mathcal{L}$ which has a point in the intersection is entirely contained in each $\mathcal{L}_i$, so is entirely contained in their intersection). Therefore, by Zorn's Lemma, this collection contains minimal elements, i.e., non-empty sublaminations properly containing no others. ∎

Lemma 5: If $\mathcal{L}'$ is an essential lamination obtained from the essential lamination $\mathcal{L}$ by splitting $\mathcal{L}$ open (see [G-O]) along some finite collection of leaves, and if $\mathcal{L}'$ contains a unique minimal sublamination, then $\mathcal{L}$ contains a unique minimal sublamination.

Proof: From the definition of splitting it follows that there is a continuous surjection $\pi : \mathrm{M} \rightarrow \mathrm{M}$ with $\pi(\mathcal{L}')$=$\mathcal{L}$ which carries leaves of $\mathcal{L}'$ onto leaves of $\mathcal{L}$ (just crush the I-fibers of the splitting regions to points).

If $\mathcal{L}_0$ is a non-empty minimal sublamination of $\mathcal{L}$, then $\mathcal{L}'_0 = \pi^{-1}(\mathcal{L}_0) \cap \mathcal{L}'$ is a sublamination of $\mathcal{L}'$. By assumption, $\mathcal{L}'$ contains a unique minimal sublamination C$'$, so because there is a minimal sublamination of $\mathcal{L}'$ in $\mathcal{L}'_0$ (there is a minimal sublamination <u>for</u> $\mathcal{L}'_0$, thought of on its own, which must then be minimal for $\mathcal{L}'$, as well), it follows that C$' \subseteq \mathcal{L}'_0$.

But then $\pi(\mathrm{C}') \subseteq \pi(\mathcal{L}'_0) = \mathcal{L}_0$ is closed in M (C$'$ is closed hence compact in M, so $\pi(\mathrm{C}')$ is compact hence closed in M) and saturated (leaves go to leaves under π), so is a non-empty sublamination of $\mathcal{L}$ contained in $\mathcal{L}_0$; therefore, $\mathcal{L}_0 = \pi(\mathrm{C}')$ by the minimality of $\mathcal{L}_0$. Therefore <u>every</u> minimal sublamination of $\mathcal{L}$ is equal to $\pi(\mathrm{C}')$, i.e., there is only one.

∎

Now let $\mathcal{L} \subseteq$M be an essential lamination carried with full support by an essential branched surface B$\subseteq$M, with M compact.

Proposition 6: If M is non-Haken, then M$\backslash$int(N(B)) consists of handlebodies; in particular, B is connected.

Proof: By the above proposition, if some component M_0 is not a handlebody, then it contains a closed incompressible surface F$\subseteq M_0$. The branched surface B$\cup$F then carries the lamination $\mathcal{L}\cup$F with full support, and it is easy to verify that B$\cup$F is essential in M. The surface F misses the branch locus of B$\cup$F, so B$\cup$F has no disks of contact or monogons because B doesn't; ∂_hN(B$\cup$F) is incompressible by the choice of F, and has no disk or sphere components because F is not a disk or sphere; and M$\backslash$N(B$\cup$F) is irreducible, again by choice of F. Finally, B$\cup$F has no Reeb branched surface: since B doesn't have one, any such would have to include F. But since any lamination carried by F must contain a surface homeomorphic to F (the first leaf you meet falling in along an I-fiber must be a 1-fold cover of a component of ∂N(F)=F$\cup$F), F must be a torus, and the Reeb component must consist of a surface parallel to F together with planar leaves limiting on it from parts of B. But this is impossible, because this would require branch curves along F, which do not exist.

Therefore by [G-O] every leaf of $\mathcal{L}\cup$F, and F in particular, is π_1-injective in M, making M Haken (it is already irreducible by [G-O]). But this contradicts our hypothesis, so every component M_0 is a handlebody. It follows that B is connected; if it weren't, then some component of M$\backslash$int(N(B)) would have more than one ∂-component. ∎

Essential laminations carried by branched surfaces with the above property are studied in [H-O], where they are called 'full'; this result therefore says that every essential lamination in a non-Haken 3-manifold is full.

Proposition 7: M and $\mathcal{L}$ as above. Then for any two leaves L_1, L_2 of $\mathcal{L}$, the intersection of their closures, $\overline{L_1} \cap \overline{L_2}$ is non-empty.

Proof: Consider the essential lamination $\mathcal{L}_0 = \overline{L_1} \cup \overline{L_2}$, carried (see [G-O]) with full support by some essential branched surface B_0. Split $\mathcal{L}_0$ open along L_1 and L_2; call the resulting lamination $\mathcal{L}'$. $\mathcal{L}'$ is still carried by B_0, and there is a canonical projection π:M$\rightarrow$M with $\pi(\mathcal{L}')=\mathcal{L}_0$ which takes an I-fiber of B_0 to itself. Let A_k, for k=1,2, be the inverse image, in $\mathcal{L}'$, of the leaf L_k, under π. Each consists of one or two leaves, bounding an I-bundle component of M$|\mathcal{L}'$, and each leaf maps onto L_k under π. The essential lamination $\mathcal{L}'$ meets the I-fibers of B_0 in nowhere-dense sets; this is because $L_1 \cup L_2$ is dense in $\mathcal{L}_0$, and (in an I-fiber) these points have been replaced by intervals meeting $\mathcal{L}'$ only in their endpoints.

By [G-O], the branched surface B_0 is <u>infinitely</u> <u>splittable</u> to $\mathcal{L}'$; there is a sequence $B_0, B_1, B_2, \ldots$ of essential branched surfaces carrying $\mathcal{L}'$ (with full support), with $\mathcal{L}' \subseteq N(B_i) \subseteq N(B_{i-1})$, and $\cap N(B_i) = \mathcal{L}'$. Applying Proposition 6 to B_i, we can then conclude that it is connected. Since for each of the leaves A_k, k=1,2, the support (in B_i) of the sublamination $\overline{A_k}$ is a (closed) sub-branched surface of B_i, it follows that these supports are not disjoint (otherwise they exhibit B_i as the union of two disjoint closed subsets). So there is an I-fiber α_i of $N(B_i)$ which meets both $\overline{A_1}$ and $\overline{A_2}$. In particular, α_i meets A_1 and A_2, in points x_i and y_i, respectively.

Consider the set of points $x_i \subseteq A_1$; this sequence has a limit point x in $L \subseteq \overline{A_1}$. But because $\mathcal{L}'$ met transverse arcs in nowhere-dense sets, it follows that the I-fibers of the

$N(B_i)$ must be becoming uniformly short (the I-fibers of the $N(B_i)$ are nested in the I-fibers of $N(B_{i-1})$, so their lengths must tend to zero, because $\mathcal{L}'$ contains no transverse arcs). Therefore, the distance between x_i and y_i must be tending to zero; but then since the sequence x_i tends to x, it follows that the sequence y_i also tends to x. But the y_i are all in A_2, and so any limit point they have lies in $\overline{A_2}$, and therefore x lies in $\overline{A_2}$, i.e., $x \in \overline{A_1} \cap \overline{A_2}$, which is therefore non-empty. But because $\pi(A_k)=L_k$ (so $A_k \subseteq \pi^{-1}(L_k)$), it then follows that $\emptyset \neq \overline{A_1} \cap \overline{A_2} \subseteq \pi^{-1}(\overline{L}_1) \cap (\pi^{-1}(\overline{L}_2)) = \pi^{-1}(\overline{L}_1 \cap \overline{L}_2)$, so $\overline{L_1} \cap \overline{L_2} \neq \emptyset$ in $\mathcal{L}_0$. ∎

This now allows us to finish the proofs of the theorem and corollary. First, to unify them, if $\mathcal{L}$ is actually a foliation $\mathcal{F}$ without Reeb components, split it open along a (finite) collection of leaves to make it an essential lamination $\mathcal{L}$ (i.e., so that it is carried by a branched surface). Now by Proposition 7, the closure of any two leaves intersect. Let $\mathcal{L}_0$ be a minimal sublamination for $\mathcal{L}$, and write it as $\mathcal{L}_0 = \overline{L_0}$ for L_0 a leaf of $\mathcal{L}_0$ ($\overline{L_0}$ is a sublamination of $\mathcal{L}_0$, and so equals $\mathcal{L}_0$ by minimality). Then for any leaf L of $\mathcal{L}$, $\overline{L} \cap \overline{L_0} = \overline{L} \cap \mathcal{L}_0$ is a non-empty sublamination of $\mathcal{L}_0$ (it is easy to see that it is closed and saturated), and therefore by minimality it equals $\mathcal{L}_0$, i.e., $\mathcal{L}_0 \subseteq \overline{L}$ for every leaf L of $\mathcal{L}$. But if $\mathcal{L}_1 = \overline{L_1}$ is any other minimal sublamination for $\mathcal{L}$, then the fact that $\overline{L_0} \cap \overline{L_1}$ is non-empty implies that $\mathcal{L}_0 = \overline{L_0} \cap \overline{L_1} = \mathcal{L}_1$, so there is only one non-empty minimal sublamination. Therefore, by Lemma 5, our original lamination (or foliation) contains a unique minimal sublamination.

Finally, we finish with a slight improvement on the result $\mathcal{L}_0 \subseteq \overline{L}$:

Corollary 8: $\mathcal{L}$, M as above, with unique minimal sublamination $\mathcal{L}_0$. Let L be a leaf of $\mathcal{L}$ having an end ϵ (see [Ni]). Then the limit set of the end, $\lim_\epsilon(L)$, contains $\mathcal{L}_0$.

Proof: $\lim_\epsilon(L)$ is a closed, non-empty, saturated subset of $\mathcal{L}$, i.e., a sublamination. It therefore contains $\mathcal{L}_0$. ∎

In other words, not only does every leaf of $\mathcal{L}$ limit on $\mathcal{L}_0$, but every end of every leaf does, as well.

2. Haken coverings of non-Haken 3-manifolds

The structure theorem above shows that the leaves of essential laminations in non-Haken 3-manifolds behave in a qualitatively different fashion, in general, from those of essential laminations in Haken manifolds. It therefore gives, in principle, a way to distinguish Haken manifolds from non-Haken ones: if a manifold contains an essential lamination which fails to have the property stated in the theorem, then the manifold must be Haken.

This distinction is potentially useful in determining when a laminar 3-manifold is virtually Haken. If M is non-Haken and contains an essential lamination $\mathcal{L}$, which therefore has the property that the closures of any two leaves intersect, and has a finite covering $\pi:\widetilde{M}\to M$ such that the inverse image $\mathcal{L}'$ of $\mathcal{L}$, which is an essential lamination in $\widetilde{M}$, has two leaves with disjoint closures, then $\widetilde{M}$ must be Haken, and so M is virtually Haken. This is a nice image; however, that is all it is:

Theorem 9: If $\pi:\widetilde{M}\to M$ is a finite covering, with M non-Haken and $\mathcal{L}\subseteq M$ an essential lamination with $\mathcal{L}_0\subseteq\mathcal{L}$ its unique minimal sublamination, then $\mathcal{L}_0'=\pi^{-1}(\mathcal{L}_0)\subseteq\pi^{-1}(\mathcal{L})=\mathcal{L}'$ is the unique minimal sublamination of $\mathcal{L}'$; consequently, any two leaves of $\mathcal{L}'$ have intersecting closures.

This theorem therefore says that an essential lamination in a non-Haken 3-manifold is *really* tightly wrapped; it can't be unwrapped by passing to finite covers of M.

The only ingredient of the proof which we do not already have is the following result:

Proposition 10: If π, $\widetilde{\mathrm{M}}$, M, $\mathcal{L}_0$, and $\mathcal{L}_0'$ are as above, and if L_0', L' are leaves of $\mathcal{L}_0'$ with $\overline{\mathrm{L}_0'}\subseteq\overline{\mathrm{L}'}$, then $\overline{\mathrm{L}_0'}=\overline{\mathrm{L}'}$.

Proof: Suppose that $\overline{\mathrm{L}_0'}\neq\overline{\mathrm{L}'}$, so that $\overline{\mathrm{L}_0'}$ is properly contained in $\overline{\mathrm{L}'}$; in particular, $\mathrm{L}'\cap\overline{\mathrm{L}_0'}=\emptyset$. We will show that $\pi(\overline{\mathrm{L}_0'})\neq\mathcal{L}_0$, which is a contradiction, because $\pi|_{\overline{\mathrm{L}_0'}}:\mathrm{L}_0'\to\mathrm{L}_0$, where L_0 is the leaf of $\mathcal{L}_0$ which L_0' maps to, is a covering map, hence onto, so $\mathcal{L}_0=\overline{\mathrm{L}}_0=\overline{\pi(\mathrm{L}_0')}\subseteq\pi(\overline{\mathrm{L}_0'})\subseteq\mathcal{L}_0$.

To do this, let N(B) be a fibered neighborhood of a branched surface carrying $\mathcal{L}_0$, and let $\mathrm{B}'=\pi^{-1}(\mathrm{B})$ and $\mathrm{N}(\mathrm{B}')=\pi^{-1}(\mathrm{N}(\mathrm{B}))$ (with I-fibers being carried to I-fibers under π). Pick a fiber X of N(B), and consider $\pi^{-1}(\mathrm{X})=\mathrm{X}_1\cup\ldots\cup\mathrm{X}_\mathrm{n}$. Pick a point $\mathrm{x}\in\mathcal{L}_0\cap\mathrm{X}$, and let $\{\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{k}\}=\overline{\mathrm{L}_0'}\cap\pi^{-1}(\mathrm{x})$, with $\mathrm{x}_\mathrm{i}\in\mathrm{X}_\mathrm{i}$, for all i. Because $\overline{\mathrm{L}_0'}$ is a closed set, for each $\mathrm{x}_\mathrm{i}\in\mathcal{L}_0'\setminus\overline{\mathrm{L}_0'}$ (i.e., each $\mathrm{i}=\mathrm{k}+1,\ldots,\mathrm{n}$) there is an open neighborhood $\mathcal{O}_\mathrm{i}$ of x_i in X_i which misses $\overline{\mathrm{L}_0'}$.

Consider $\mathcal{O}=\pi(\mathcal{O}_1)\cap\ldots\cap\pi(\mathcal{O}_\mathrm{k})\subseteq\mathrm{X}$. Because π maps the fibers X_i homeomorphically to X, this is an open subset of X containing x. Consider $\pi^{-1}(\mathcal{O})\subseteq\pi^{-1}(\mathrm{X})$; note that $\pi^{-1}(\mathcal{O})\cap\mathrm{X}_\mathrm{i}\subseteq\mathcal{O}_\mathrm{i}$ for $\mathrm{k}+1\leq\mathrm{i}\leq\mathrm{n}$.

Now look at $\pi^{-1}(\mathcal{O})\cap\mathrm{X}_\mathrm{k}=\mathcal{O}'$ (so $\pi(\mathcal{O}')\subseteq\mathcal{O}$). This is an open neighborhood of x_k in X_k. Because $\overline{\mathrm{L}'}$ contains ($\overline{\mathrm{L}_0'}$ and hence) x_k, there are points of L' in X_k which pass arbitrarily close to x_k and hence are contained in $\mathcal{O}'$. Choose one; call it y_k'. Then $\mathrm{y}_\mathrm{k}'\notin\overline{\mathrm{L}_0'}$, and $\pi^{-1}(\pi(\mathrm{y}_\mathrm{k}'))\cap\mathrm{X}_\mathrm{i}=\pi^{-1}(\mathrm{y}')\cap\mathrm{X}_\mathrm{i}\in\mathcal{O}_\mathrm{i}$ are not in $\overline{\mathrm{L}_0'}$ for $\mathrm{k}+1\leq\mathrm{i}\leq\mathrm{n}$. So by choosing a different point y' of $\mathcal{O}$ we have increased the number of points, in the inverse image of a point of

$\mathcal{L}_0 \cap X$, which are not in $\overline{L_0'}$. Continuing, we can therefore find a point y of $\mathcal{L}_0 \cap X$ (in fact, in $\mathcal{L}_0 \cap \mathcal{O}$) with $|\pi^{-1}(y) \cap (\mathcal{L}_0' \setminus L_0')| = n$, i.e., $\pi^{-1}(y) \subseteq \mathcal{L}_0' \setminus L_0'$. But this means that $y \notin \pi(\overline{L_0'})$, a contradiction. So $\overline{L_0'} = \overline{L'}$. ∎

The above result is a special case of a more general result on the structure of pullbacks of laminations under non-zero degree maps; see [Br1],[Br2].

Proof of Theorem 9: For technical reasons, we must first (as in Proposition 7) split $\mathcal{L}_0$ open along a leaf, to insure that $\mathcal{L}_0$ (and hence $\mathcal{L}_0'$) meets I-fibers of some (hence every) branched surface neighborhood in nowhere-dense sets. We will first prove Theorem 9 for this (possibly different) collection of laminations.

Proposition 10 implies that for any pair of leaves L, L$'$ of $\mathcal{L}_0'$, either $\overline{L} \cap \overline{L'} = \emptyset$ or $\overline{L} = \overline{L'}$. Consequently, $\mathcal{L}_0'$ is the union of a finite number of disjoint minimal sublaminations, $\overline{L}_1, \ldots, \overline{L}_n$. This is because the closure of any leaf is a minimal sublamination, and every minimal sublamination of $\mathcal{L}_0'$ maps onto $\mathcal{L}_0$ under π. Thus for a given leaf L of $\mathcal{L}_0$, every minimal sublamination of $\mathcal{L}_0'$ contains an inverse image leaf of L, of which there are only finitely many.

Claim: n=1.

Otherwise, there is a branched surface closely approximating $\mathcal{L}_0'$ which is not connected. To see this, just choose one whose fibered neighborhood has fibers with length less than δ = half of the distance between two components of $\mathcal{L}_0'$; this can be done because (as in Proposition 7) $\mathcal{L}_0'$ meets I-fibers in nowhere-dense sets). But this also means that there is an (essential) branched surface B carrying $\mathcal{L}_0$ with full support whose inverse image under π is not connected; choose one with fibers of the same short length δ.

But this situation is absurd; B has complement consisting entirely of handlebodies, hence so does $\pi^{-1}(\mathrm{B})$ (the only thing that finitely covers a handlebody is a handlebody), so every component of $\widetilde{\mathrm{M}}\backslash\mathrm{N}(\pi^{-1}(\mathrm{B}))$ has connected boundary, so $\pi^{-1}(\mathrm{B})$ is connected.

Therefore $\mathcal{L}'_0$ is a minimal sublamination of $\mathcal{L}'$. But then since for every leaf L' of $\mathcal{L}'$, $\overline{\pi(\mathrm{L}')}\subseteq\pi(\overline{\mathrm{L}'})$, and since $\mathcal{L}_0\subseteq\overline{\mathrm{L}}=\overline{\pi(\mathrm{L}')}$ for $\pi(\mathrm{L}')\subseteq\mathrm{L}\subseteq\mathcal{L}$ (since $\pi(\mathrm{L}')=\mathrm{L}$), we have that $\pi(\overline{\mathrm{L}'})\cap\mathcal{L}_0\neq\emptyset$, so $\overline{\mathrm{L}'}\cap\pi^{-1}(\mathcal{L}_0)=\overline{\mathrm{L}'}\cap\mathcal{L}'_0\neq\emptyset$. Therefore $\mathcal{L}'_0\subseteq\overline{\mathrm{L}'}$. So any two leaves of $\mathcal{L}'$ have closures whose intersection contains $\mathcal{L}'_0$; it is therefore the unique minimal sublamination of $\mathcal{L}'$.

But now if we collapse our split open leaves of $\mathcal{L}'_0$ back again, Lemma 5 then insures that $\mathcal{L}'$ contains a unique minimal sublamination; in particular, it is the image under collapsing of the minimal sublamination $\mathcal{L}'_0$, i.e., it is the original $\mathcal{L}'_0$. ∎

As a final note, we should point out that not only is this true for $\pi^{-1}(\mathcal{L})=\mathcal{L}'$, but also for any sublamination $\mathcal{L}'_1$ of $\mathcal{L}'$; the closure of any leaf of $\mathcal{L}'_1$ is the same whether we think we are in $\mathcal{L}'_1$ or $\mathcal{L}'$, so it contains $\mathcal{L}'_0$. In particular, any essential branched surface carrying $\mathcal{L}'_1$ has complement consisting of handlebodies. For otherwise (by the proof of Proposition 6) there would be an incompressible surface in $\widetilde{\mathrm{M}}$ missing $\mathcal{L}'_1$, hence missing $\mathcal{L}'_0$, hence contained in the handlebody complement of some essential branched surface carrying $\mathcal{L}'_0$, a contradiction.

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AMS Subject Classifications: 57M10, 57M99

Keywords: essential lamination, non-Haken, virtually Haken