Graph manifolds and taut foliations

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Abstract. We examine the existence of foliations without Reeb components, taut foliations, and foliations with no $S^1 \times S^1$-leaves, among graph manifolds. We show that each condition is strictly stronger than its predecessor(s), in the strongest possible sense; there are manifolds admitting foliations of each type which do not admit foliations of the succeeding type(s).

§0
Introduction

Taut foliations have been increasingly useful in understanding the topology of 3-manifolds, thanks largely to the work of David Gabai [Ga1]. Many 3-manifolds admit taut foliations [Ro1],[De],[Na1], although some do not [Br1],[Cl]. To date, however, there are no adequate necessary or sufficient conditions for a manifold to admit a taut foliation. This paper seeks to add to this confusion.

In this paper we study the existence of taut foliations and various refinements, among graph manifolds. What we show is that there are many graph manifolds which admit foliations that are as refined as we choose, but which do not admit foliations admitting any further refinements. For example, we find manifolds which admit foliations without Reeb components, but no taut foliations. We also find manifolds admitting $C^{(0)}$ foliations with no compact leaves, but which do not admit any $C^{(2)}$ such foliations. These results point to the subtle nature behind both topological and analytical assumptions when dealing with foliations.

A principal motivation for this work came from a particularly interesting example; the manifold M obtained by 37/2 Dehn surgery on the $(-2,3,7)$ pretzel knot K. This manifold is a graph manifold, obtained by gluing two trefoil knot exteriors together along their boundary tori. We show that every essential lamination in M contains a torus leaf, and therefore every essential lamination intersects the image of K in M. This tells us a great deal about essential laminations in the exterior of K. This is discussed in Section 5 below.

The paper is organised as follows. In Section 1 we give the necessary background on Seifert-fibered spaces and graph manifolds, and introduce the appropriate numerical coordinates for describing them. In Section 2 we gather the relevant results

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on foliations and essential laminations to carry out our proofs. Section 3 gives the main reults of the paper, and Section 4 provides the proofs. Section 5 discusses surgery on the $(-2,3,7)$ pretzel knot. Section 6 finishes with some speculations.

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§1
Coordinates for graph manifolds

A Seifert-fibered space M is an S^1-bundle whose base is a 2-orbifold. More precisely, a Seifert-fibered space begins with an honest circle bundle M_0 over a compact surface; for our purposes it will suffice to think about a compact, orientable, surface, possibly with boundary, crossed with S^1. To some of the boundary components of M_0 we then glue a collection of solid tori, so that the meridional direction of each solid torus does <u>not</u> correspond to the S^1-direction on the boundary of M_0. The induced foliation of the boundaries of each of these solid tori by circles extends, in an essentially unique way, to a foliation by circles of the solid torus, so that the core of the solid torus is a leaf. This gives a foliation of M by circles, whose space of leaves - the quotient space obtained by crushing each circle leaf to a point - is a 2-orbifold. Its underlying topological space is called the base surface of the Seifert-fibering of M. The cone points of the orbifold correspond to the cores of the solid tori; these cores are called the *multiple fibers* of the Seifert-fibering of M.

A manifold M is a graph manifold if it there is a collection $\mathcal{T}$ of disjoint embedded tori so that the manifold $M|\mathcal{T}$ obtained by splitting M open along $\mathcal{T}$ is a (not necessarily connected) Seifert-fibered space. We assume that the collection $\mathcal{T}$ is minimal, in the sense that for no torus T in $\mathcal{T}$ is $M|(\mathcal{T}\backslash T)$ a Seifert-fibered space. We adopt the convention that a Seifert-fibered space is not a graph manifold, so $\mathcal{T}\neq\emptyset$. Since the fibering of a Seifert-fibered space is essentially unique [S], we can give a more constructive approach to minimality. Thinking in reverse, a graph manifold is obtained by gluing Sefiert-fibered spaces together along some of their boundary tori; the glued tori become the collection of splitting tori $\mathcal{T}$. The collection $\mathcal{T}$ is minimal if, in gluing, the homotopy class of the circle fiber in one boundary torus is not identified with the class of the fiber in the other boundary torus. The only exceptions to this rule occur when some components are solid tori or $T\times I$; for solid tori, minimality requires that the meridion in the boundary of the solid torus be glued to the S^1-fiber, and a $T\times I$ can either be absorbed into a component of M_0 (if its ends are not glued together), or must have its ends glued together by a map having (on the level of $H_1(T;\mathbb{R})$) no integer-valued eigenvectors.

Our results will be stated in terms of the Seifert-fibered pieces making up the graph manifold, and the gluing maps between their boundary tori. To do so, we will need a proper set of coordinates.

In [S] Seifert developed numerical invariants of what he called 'fibered spaces', and gave a complete classification of them in terms of these invariants. They describe the topological type of the base orbifold, and the way that the the regular fibers spin around the multiple fibers. More explicitly, an orientable Seifert-fibered space M can be described as follows: start with a compact surface F of genus g and b boundary components (the underlying topological space of the base orbifold), and drill out k disks (one for each multiple fiber of the Seifert-fibering). To be sure the resulting surface has non-empty boundary, drill out one more 'zero-th' disk, giving a surface F_0. Now construct the (unique) S^1-bundle M_0 over F_0 with orientable total space. This bundle has a (not necessarily unique) cross-section s:$F_0 \rightarrow M_0$ (because $\partial M_0 \neq \emptyset$). The images of ∂F_0 in ∂M_0, together with the circle fibers in ∂M_0, give us a system of coordinates for curves in ∂M_0, defining for each simple closed curve in a component of ∂M_0 a slope in $\mathbb{Q} \cup \{\infty\}$, where the section defines slope 0 and the fiber defines slope ∞. We then glue $k+1$ solid tori back onto M_0 to obtain M. The gluing of the i-th solid torus identifies the boundary of a meridion disk to some curve a_i(fiber) + b_i(section) in ∂M_0. These gluings completely describe the Seifert-fibered space, giving us its so-called Seifert invariant

$$M = \Sigma(\pm \text{ g,b}; a_0/b_0,a_1/b_1,\ldots,a_k/b_k) .$$

$\pm$ equals + if F is orientable, $-$ if not. The rational numbers a_i/b_i are treated as an unordered $(k+1)$-tuple. The denomenator of each rational number (in lowest terms) turns out to be the multiplicity of the corresponding multiple fiber. Since our zero-th disk did not correspond to a multiple fiber, its multiplicity is 1, so b_0=1.

This invariant is dependent upon the choice of section for M_0; the only way this section can change, however, is by summing along vertical annuli and tori (see Figure 1). Summing along a torus does not change the associated invariant, and summing along an annulus changes the invariant in a very controlled way; it adds and subtracts 1 each from the invariants associated to the two components of ∂M_0 containing the boundary of the annulus.

Figure 1

We can actually remove this ambiguity by exploiting it; by a series of summings along annuli one of whose boundaries lies over the boundary of the zero-th disk, we can arrange that $0 \leq a_i/b_i < 1$, for every $i=1,\ldots,k$; essentially, this amounts to gathering the integer parts of the a_i/b_i into $a_0/b_0 = a_0$. This gives us a *normalized* Seifert invariant

$$M = \Sigma(\pm \text{ g,b}; a_0/1,a_1/b_1,\ldots,a_k/b_k)$$

where $a_i,b_i \in \mathbb{Z}$, and $0 < a_i < b_i$, for $i=1,\ldots,k$. Seifert showed that a Seifert-fibered space is determined up to orientation- and fiber-preserving homeomorphism by its normalized Seifert-invariant. The normalized Seifert-invariant for M with the opposite orientation is

$$M = \Sigma(\pm \text{ g,b}; (-a_0-k)/1,1-(a_1/b_1),\ldots,1-(a_k/b_k)).$$

If M has non-empty boundary (b$\neq$0), we can sum along annuli one of whose components is over the zero-th disk and the other in ∂M, to make a_0=0; this means that the boundary of the meridion disk is glued to the boundary of the section, allowing us to extend the section over the zero-th solid torus, and absorbing the solid torus into the circle bundle without losing a section. In this case we can, if we wish, delete a_0 from the normalized invariant.

We must also be able to describe the gluings from which we build our graph manifolds out of their Seifert-fibered pieces. A homeomorphism between two 2-tori is determined by its action on first homology $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$, and is therefore given by an element of $SL_2(\mathbb{Z})$, once bases for the first homology of the two tori have been fixed. We will use as our bases for $H_1(T)$ the section-fiber pairs that we have described above. If a Seifert-fibered piece has more than one boundary component, there is still some freedom in the choice of section; when this occurs, we will simply choose one best suited to our needs at that time.

In what follows, we will for notational convenience let 'A = (a,b;c,d)' denote the matrix

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Simple closed curves γ in our boundary tori are represented by $\mathbb{Z}$-linear combinations $\alpha\mathbf{f}+\beta\mathbf{s} = (\alpha,\beta)\in\mathbb{Z}\oplus\mathbb{Z}$ of our fiber/section basis, with α and β relatively prime; we therefore often think of γ as being represented by the rational number α/β. A homeomorphism of boundary tori, represented by the matrix A$=(a,b;c,d)$, sends the curve $\gamma=(\alpha,\beta)$ to the curve

$$\mathrm{A}(\alpha,\beta) = (a\alpha+b\beta,c\alpha+d\beta) \leftrightarrow \frac{a\alpha+b\beta}{c\alpha+d\beta} = \frac{a\frac{\alpha}{\beta}+b}{c\frac{\alpha}{\beta}+d}$$

and so thought of as a map A:$\mathbb{Q}\rightarrow\mathbb{Q}$, it is the map

$$\mathrm{A}(x)=\frac{ax+b}{cx+d}.$$

This function A extends naturally over $\mathbb{R}$, where it describes the effect of the homeomorphism A on the slopes of irrational foliations of the torus, as well. It has derivative $A'(x)=\dfrac{(ad-bc)}{(cx+d)^2}$, which always has the same sign (which depends upon $ad-bc=\det(A)$), so A maps any interval (x_1,x_2) not containing the asymptote $-d/c$ of A montonically to either $(A(x_1),A(x_2))$ or its reverse.

§2
Taut foliations, essential laminations, and the like

The reader is referred to [Ga2],[Ga3],[G-O] for basic notions on taut foliations and essential laminations. A codimension-one foliation $\mathcal{F}$ of a 3-manifold M has no Reeb components if no leaf of $\mathcal{F}$ is a compressible torus. The strongest known necessary condition for a 3-manifold M to admit a foliation without Reeb components is that its universal cover be homeomorphic to $\mathbb{R}^3$ [Pa]. A foliation $\mathcal{F}$ is *taut* if every leaf has a closed loop passing through it which is everywhere transverse to the leaves of $\mathcal{F}$. Taut foliations have no Reeb components. It is an important result of Goodman [Go] that if a foliation is not taut then it contains a (not necessarily compressible) torus leaf. Therefore, foliations with no torus leaves are taut. Finally, if a 3-manifold admits an Anosov flow (see, e.g., [Fe1]), then the stable foliation of the flow is a codimension-one foliation whose leaves are (open) planes, annuli, and Möbius bands. In particular, the foliation has no compact leaves, and hence no torus leaves. Essential laminations generalize the notion of a foliation without Reeb components to 'partial' foliations, which fill up a closed subset of a 3-manifold M, and provide a convenient framework in which to discuss the sructure of foliations.

Our constructions rely on two main points. Every essential lamination (and therefore every taut foliation) in a Seifert-fibered space M contains a sublamination which is either horizontal (its leaves are everywhere transverse to the circle fibers of M) or vertical (its leaves are foliated by fibers of M). Also, most (closed) Seifert-fibered spaces do not contain horizontal laminations. Details are given in the propositions gathered together below.

Proposition 1 [Th],[Le]. *If M admits a taut foliation $\mathcal{F}$, and T is an incompressible torus in M, then T may be isotoped either to be everywhere transverse to $\mathcal{F}$, or to be a leaf of $\mathcal{F}$.*

If T is not isotopic to a leaf of $\mathcal{F}$, then after making T transverse to $\mathcal{F}$, $\mathcal{F}|T{=}\mathcal{F}_0$ may not be an 'essential' foliation in $M|T{=}M_0$. The obstruction is a 'half-Reeb' component: a leaf of $\mathcal{F}_0$ which is a boundary-parallel annulus (Figure 2). But half-Reeb components can be eliminated by a further isotopy of T; this is most easily seen by the following minimal surface argument, due to Joel Hass. In [Ha], Hass shows that $\mathcal{F}$ and T can be isotoped so that the leaves of $\mathcal{F}$, and T, are minimal surfaces in M. This immediately implies that T is transverse to the leaves of $\mathcal{F}$. But it also follows that $\mathcal{F}|T$ has no half-Reeb components, since the annulus leaf of $\mathcal{F}|T$ is isotopic rel boundary to the obvious annulus in T. Since minimal surfaces are area minimizing over compact sets, the two annuli have the same area, but then swapping them and rounding corners reduces the area of the torus T while remaining in the same isotopy class, for example, a contradiction.

This result has also been extended to essential laminations by Roberts [Ro2].

Figure 2

Proposition 2 [Br2]. *Let M be an orientable Seifert-fibered space with non-empty boundary, which does not contain a horizontal annulus, and let $\mathcal{L}$ be an essential lamination in M, meeting ∂M transversely in a lamination $\mathcal{L}\cap\partial M$ containing a Reeb-foliated annulus. Then $\mathcal{L}$ contains a vertical sublamination, which intersects ∂M. In particular, the Reeb annulus is vertical.*

This means that usually a taut foliation in a Seifert-fibered space meets boundary tori in suspensions. In particular, in the cases we will be consideringm where T is a splitting torus of a graph manifold, the foliations $\mathcal{F}|T$ above will meet T in suspensions. For otherwise they contain vertical sublaminations on both sides of T, so the gluing map A has glued the circle fiber on one side to the circle fiber on the other, so M is again Seifert-fibered, contradicting the minimality of $\mathcal{T}$.

Proposition 3a [Br1],[Br2]. *Every essential lamination in a Sefiert-fibered space (with or without boundary) contains a vertical or horizontal sublamination.*

Proposition 3b [Br1],[Br2]. *Every essential lamination in a Seifert-fibered space M, whose base orbifold B is S^2 with three cone points, containing no (horizontal) torus leaves, is horizontal. Every essential lamination in a Seifert-fibered space M, with base orbifold $\mathbb{D}^2$ and two cone points, containing no (horizontal or vertical) annuli or (vertical, hence ∂-parallel) tori, is horizontal.*

Proposition 3c [Br2]. *If M is a Seifert-fibered space with boundary, which contains no horzontal annuli, then every essential lamination which does not contain a vertical sublamination is isotopic to a horizontal lamination.*

In particular, for Seifert-fibered knot exteriors (i.e., torus knots), we have:

Proposition 3d [Br2]. *Every essential lamination in the exterior of a torus knot either contains the (vertical) cabling annulus (or Möbius band) as a leaf or is isotopic to a horizontal lamination.*

Proposition 4 [Sa]. *If $\mathcal{F}$ is a $C^{(2)}$ foliation of a connected manifold M, and $\mathcal{L}$ is a minimal set of $\mathcal{F}$ consisting of more than one leaf and such that each leaf of $\mathcal{L}$ has trivial linear holonomy, then $\mathcal{L} = \mathcal{F}$.*

A minimal set is a sublamination $\mathcal{L}$ so that every leaf of $\mathcal{L}$ has closure (in M) $\mathcal{L}$. Holonomy is the (germ at 0 of the) injective map between subintervals of $[-1,1]$ obtained by looking at how the leaves of a foliation $\mathcal{F}$ intersect a small annular fence lying over a closed loop in a leaf of $\mathcal{F}$ (Figure 3). Linear holonomy is the derivative at 0 of this map. A leaf L of $\mathcal{F}$ has trivial (linear) holonomy if for every loop in L the induced map (or its derivative) is the identity (or 1). Note that linear holonomy only makes sense for foliations with smooth transverse structure.

Figure 3

Finally, the most important facts we will use restrict the Seifert-fibered spaces which can admit horizontal foliations. Let M be an orientable Seifert-fibered space with orientable base orbifold and normalized Seifert invariant

$$\Sigma(\text{g},\text{b};\ a_0/1, a_1/b_1, \ldots, a_k/b_k).$$

As our motivating example, let F=a 2-sphere with k punctures, and M=$F \times S^1$ (so M has Seifert invariant $\Sigma(0,k;0)$). Label the components of ∂M by $\{1,\ldots,k\}$. Suppose M admits a horizontal foliation $\mathcal{F}$, meeting some subset J$\subseteq\{1,\ldots,k\}$of the boundary components in parallel loops or a foliation with no compact leaves, and meeting the other boundary components in more complicated, Kronecker-type foliations. The induced foliations of the boundary components of M can be assigned a 'slope' γ_i, after coordinates are given for each torus; it is essentially the rotation number of the return map given by following points on the slope ∞ curve around the leaves of the induced foliation, until they return to the slope ∞ curve again. Choose a section of $M\backslash\text{int}(N(\text{regular fiber}))$ so that all of these slopes γ_i lie in [0,1). Relative to this section, M then has Seifert invariant $\Sigma(0,k;\ a_0)$ for some integer a_0.

Proposition 5 [JN],[Na]. *Let M be as above. Then $\mathcal{F}$ exists if and only if either some number d of the γ_i=0 and $2-k \leq a_0 \leq d-2$, or $a_0=-1$ or -2, and for some integers $1\leq c < m$ and some permutation*

$$\frac{c_1}{m},\ldots,\frac{c_n}{m} \text{ of } \frac{a}{m},\frac{m-a}{m},\frac{1}{m},\ldots,\frac{1}{m}, \text{ we have}$$

($a_1=-1$) (*) $\gamma_i < \dfrac{c_i}{m}$ if $i\in J$, and $\gamma_i \leq \dfrac{c_i}{m}$ if $i \notin J$, or

($a_1=-2$) After replacing each γ_i by $1-\gamma_i$, (*) holds.

We note in passing that Proposition 5 immediately implies (by starting with a horizontal foliation and then drilling out neighborhoods of the multiple fibers - in this case J={1,2,3}):

Proposition 6 [JN],[Na]. *If g=0, b=0, and k=3, then M admits a horizontal foliation if and only if $a_0=-1$ or -2, and*

(a_0=-1) there exists integers 1$\leq a < m$ such that, up to permutation,

$$(*)\ \frac{a_1}{b_1} < \frac{a}{m},\ \frac{a_2}{b_2} < \frac{m-a}{m},\ \text{and}\ \frac{a_3}{b_3} < \frac{1}{m},\ \text{or}$$

(a_0=-2) the same condition (*) holds with each $a_i//bi$ replaced by 1-(a_i/b_i).

Similar conditions can be formulated for k>3, see [JN].

This result, together with [Br1],[Cl], provided the first examples of 3-manifolds with universal cover $\mathbb{R}^3$ which do not admit any foliations without Reeb components.

Similar results also hold for manifolds with higher genus base orbifold:

Proposition 7 [EHN]. *If g>0, and b=0 (i.e. ∂M=$\emptyset$), then M admits a horizontal foliation if and only if (2-2g)-k$\leq a_1 \leq$(2-2g)*

As with the genus 0 case, there is an analogous statement for Seifert-fibered spaces with boundary. We will only need the following special case:

Proposition 8 [EHN]. *Every horizontal foliation in the Seifert-fibered space M = Σ(1,1; 0) (i.e., M=(a once-punctured torus)$\times S^1$) meets the boundary torus in a foliation of slope $\gamma\in$[0,1). Furthermore, all slopes in [0,1) are realized by horizontal foliations (and can be assumed to meet ∂M in a 'linear' foliation of ∂M, if $\gamma\neq$0).*

§3
The results

What we will now show is that, under appropriate conditions, a taut foliation must meet each Seifert-fibered piece of certain graph manifolds in horizontal foliations. Propositions 5 and 6, suitably applied, then yield restrictions on the gluings which would allow horizontal foliations to match up, for our foliation to exist in the first place.

Almost every graph manifold admits codimension-one foliations without Reeb components; if none of the components of M_0 is a solid torus, then taking a horizontal foliation on each component of M_0 and 'spinning' them (see Figure 4) as they approach ∂M_0, we get a foliation without Reeb components on M, having the tori $\mathcal{T}$ as leaves. However, nothing else comes for free.

Figure 4

Theorem A [Br1],[Cl]. *There exist infinitely many Seifert-fibered spaces with universal cover $\mathbb{R}^3$ which admit no foliations without Reeb components.*

Theorem B. *There exist infinitely-many graph manifolds M which admit no taut foliations - every foliation contains a separating torus leaf.*

Theorem C. *There exist three Seifert-fibered spaces M which admit taut foliations, but each must have a (non-separating) torus leaf.*

Theorem D. *There exist infinitely-many graph manifolds M which admit $C^{(0)}$ taut foliations with no compact leaf, but every $C^{(2)}$ foliation must have a torus leaf.*

Theorem E. *There exist infinitely-many graph manifolds M which admit $C^{(0)}$ taut foliations with no compact leaf, but no Anosov flows.*

What we in fact show is that the torus leaves that must necessarily exist are the gluing tori used to build the graph manifold from its Seifert-fibered pieces. We shall prove each case separately; each requires studying the existence of taut foliations on a different class of graph manifolds. Basically, by changing the toplogical type of the base orbifolds of the Seifert-fibered pieces, we will guarantee the existence of each type of foliation, while avoiding the existence of their stronger cousins.

The only one of these theorems which really is unsatisfying is Theorem C; the examples we provide might be thought of as a coincidence. It is possible that many examples can be found among the graph manifolds obtained from a Seifert-fibered space over the annulus, with one multiple fiber, by gluing its two boundary components together. The analysis of slopes of horizontal foliations over the annulus must be refined, however; this will be addressed in a later paper.

§4
The proofs

Theorem B:

In this case we use a graph manifold M consisting of two Seifert-fibered spaces M_0, M_1 with base D^2 and two multiple fibers, glued together along their boundaries. These have normalized Seifert invariants

$$\Sigma(0,1;\, \gamma_1,\gamma_2) \text{ and } \Sigma(0,1;\gamma_1',\gamma_2')$$

The fibering of each piece is unique, and therefore the manifold resulting from gluing the two togther will be Seifert-fibered only if the gluing map preserves the fiber direction on each torus; on the level of matrices, this means that the gluing map is a 'shear' $A = (\pm 1, n; 0, 1)$.

Let $\mathcal{F}$ be a taut foliation on M. No leaf of $\mathcal{F}$ can be isotopic to the torus splitting M into its Seifert-fibered pieces, because then $\mathcal{F}$ induces a foliation without Reeb components on each piece (minus a regular neighborhood of its boundary), transverse to the boundary. With the exception of $\Sigma(0,1;1/2,1/2)$, which contains a horizontal annulus, these foliations must therefore be horizontal, by Proposition 3b. But this means $\mathcal{F}$ is transversely oriented and contains a separating torus leaf, hence cannot be taut. Therefore $\mathcal{F}$ can be made transverse to the splitting torus T with no Reeb annuli, and so, again, splits to give horizontal foliations of each Seifert-fibered piece.

Using the (essentially unique) horizontal sections that allow us to define the normalized invariants of M_1 and M_2, we can assign slopes to horizontal foliations in the ∂M_i. Let $\mathcal{F}_i$ be a horizontal foliation in M_i, with boundary slope γ. By incorporating the normalizing term a_0 into γ, Proposition 5 immediately implies:

Proposition. *Let M_i be as above. Then $\mathcal{F}_i$ exists if and only if either $\gamma=-1$ or there exists integers $1\leq c < m$ and a permutation*

$$\frac{c_1}{m}, \frac{c_2}{m}, \frac{c_3}{m} \text{ of } \frac{a}{m}, \frac{m-a}{m}, \frac{1}{m} \text{ such that either}$$

$\gamma_i < \dfrac{c_i}{m}$, *for* $i=1,2$, *and* $0 \leq \gamma_3 + 1 \leq \dfrac{c_3}{m}$, *or*

$1-\gamma_i<\frac{c_i}{m}$, for i=1,2, and $0\leq -(1+\gamma_3)\leq\frac{c_3}{m}$.

Note that for the first case to be possible, we must have $\gamma_1+\gamma_2<1$, since $1/m < a/m, (m-a)/m$. Similarly, in the second case we must have $\gamma_1+\gamma_1>1$. This therefore gives us only the possibilities

$\gamma_1+\gamma_2=1$, and $\gamma=-1$ (which happens to correspond to a horizontal compact surface; the condition is that the sum of the slopes equal 0),

$\gamma_1+\gamma_2<1$, and $\gamma\in[-1,\frac{-1}{m}]\subseteq[-1,0)$ for some m (since $\gamma+1$ can be at most $(m-1)/m$), or

$\gamma_1+\gamma_2>1$, and $\gamma\in[-2+\frac{1}{m},-1]\subseteq(-2,-1]$ (since $-(\gamma+1)$ can, again, be at most $(m-1)/m$)).

Therefore, in every case, the slope of a horizontal foliation lies in $(-2,0)$. Throwing in the possibility of a vertical sublamination which intersects the boundary adds slope ∞. So to achieve our non-realizeability result, we must merely construct gluing maps A:$T\to T$ so that, on the level of boundary slopes,

(**) $A((-2,0)\cup\{\infty\})\cap((-2,0)\cup\{\infty\})=\emptyset$.

This is quite readily done; for example the map A=(0,−1;1,0), which is the map $A(x)=-1/x$, does this. With a bit of work, it is not hard to find many others.

For $A=(a,b;c,d)$, $\infty\notin A((-2,0)$ means $-d/c\notin(-2,0)$, while $A(\infty)\notin(-2,0)$ means $a/c\notin(-2,0)$. Focussing on the case that $\mathrm{Det}(A)=ad-bc=1$, we then have

$A(-2,0)=(\frac{2a-b}{2c-d},\frac{b}{d})$, so (**) requires (in addition to $ad-bc=1$)

(1) $-d/c\leq-2$ (i.e., $d/c\geq 2$) or $-d/c\geq 0$ (i.e., $d/c\leq 0$),
(2) $a/c\leq -2$ or $a/c\geq 0$, and
(3) $b/d\leq -2$ or $(2a-b)/(2c-d)\geq 0$.

The easiest way to arrange this is to make $a,d>0$, while $b,c<0$, $ad-bc=|a||d|-|b||c|=1$, and $|a|\geq 2|c|$ and $|b|\geq 2|d|$. For example,

$A=(a,b;c,d)=(n+1,-(nd+d-1);-1,d)$ with $n\geq 2$ and d$\geq$1, or

$A=(2n+1,-(2nm+m+n);-2,2m+1)$ with $n\geq 2$ and $m\geq 1$

suffice. The reader can easily supply more.

The requirement $\gamma\in(-2,0)$ is of course necessary for the existence of a horizontal foliation, but never, in fact, sufficient. Exact conditions depend upon knowing what γ_1 and γ_2 are. For example, if $\gamma_1=1/3$ and $\gamma_2=1/5$, then $0\leq\gamma+1\leq 1/2=1/m$ is possible, since $1/3<1/2=a/m$ and $1/5<1/2=(m-a)/m$. This is in fact the best possible, since it is the largest $1/m$ possible, and otherwise one of 1/3,1/5 would have to be $<1/m$, so m=2,3, or 4, and each case can be checked separately to see that it gives no better bounds. This analysis can be applied to any slopes γ_1 and γ_2 supplied; after finding one $\gamma+1\leq 1/m$ which works, one can check all smaller m's to see if a corresponding a lets $\gamma+1\leq 1/m$ work. Then one finds the largest m so that one of $\gamma_1,\gamma_2<1/m$, and checks it and all smaller m's to see for what a's does $\gamma+1\leq a/m$ work. In this way, one can find, for example, that for

$\gamma_1=1/3,\gamma_2=1/5$, then $\gamma\in[-1,-1/2]$, for
$\gamma_1=2/3,\gamma_2=1/5$, then $\gamma\in[-1,-3/4]$, for

γ_1=1/7,γ_2=1/5, then $\gamma\in[-1,-1/4]$, for
γ_1=2/7,γ_2=1/5, then $\gamma\in[-1,-1/3]$, for
γ_1=1/3,γ_2=4/5, then $\gamma\in[-5/4,-1]$, and for
γ_1=2/3,γ_2=4/5, then $\gamma\in[-3/2-1,]$.

We know, however, from [JN], that any element γ in the interior of such an interval can be realized by a horizontal foliation which meets the boundary in parallel loops of slope γ . Therefore, if the gluing map A has A(interval for first piece) meet the interval for the second piece in its interior, then we can glue two such foliations together to obtain a taut foliation. Such a foliation usually has no compact leaves; in fact, for only one γ can the foliation on a Seifert-fibered piece have a compact leaf (the one which sums with the γ_i to give 0); gluing a foliation with no compact leaves to a foliation all of whose leaves meet the boundary obviously gives a foliation with no compact leaves.

We also note that the generalization of Theorem B to essential laminations is true; every essential lamination in these manifolds contains a torus leaf. The result has the identical proof, since an essential lamination can be made transverse to the gluing torus, so that the split open pieces are essential; the split open pieces must then be horizontal, or contain a vertical annulus by Proposition 3b (again, except for Σ(0,1;1/2,1/2)). If they are horizontal, then they extend to horizontal foliations, so their slopes fall into the same restrictive range.

Theorem C:

For this case we will use the three Seifert-fibered spaces $\Sigma(0,0;-1,1/2,1/3,1/6)$, $\Sigma(0,0;-1,1/2,1/4,1/4)$, and $\Sigma(0,0;-1,1/3,1/3,1/3)$, i.e., the three Seifert-fibered spaces having base S^2 with a Euclidean orbifold structure and 3 cone points. Each of these manifolds contains a horizontal torus, so can be tautly foliated by horizontal tori. By Proposition 3b, every essential lamination in these spaces is isotopic to a horizontal one. But every horizontal lamination contains a torus leaf. Matsumoto [Ma] outlines a proof of this in the $C^{(2)}$ case, using a result of Plante [Pl] on the polynomial growth of leaves of foliations. Plante's argument is essentially $C^{(1)}$, but the only place this hypothesis is used is to show that a hypothetical foliation with no compact leaves admits no null-homotopic loops transverse to the foliation. This assertion follows easily, however, either from the fact that our foliation is horizontal (transverse loops must travel non-trivially around the fiber direction), or, more generally, from the $C^{(0)}$ proof of Novikov's theorem [So].

Theorems D and E:

In these cases we will use a graph manifold M consisting of two copies of M_i=(a once-punctured torus)$\times S^1$ i=1,2, glued together along their boundaries. They both have normalized Seifert invariant

$$\Sigma(1,1;\ 0)$$

Again, the fibering on each piece is unique, so the resulting manifold is Seifert-fibered if and only if the gluing map A is a shear. We therefore assume that A is not a shear.

For all gluings A, the resulting manifold contains a $C^{(0)}$ foliation with no compact leaves. The foliation has three parts. In each piece M_i we put a vertical lamination $\mathcal{L}_i=\lambda_i\times S^1$, where λ_i is a 1-dimensional lamination in the once-punctured torus, with no compact leaves, and having every leaf dense in λ_i. Such (measured) laminations are easily built carried by the standard train track in the 2-torus (Figure 5a). For every gluing A, $M\backslash(\mathcal{L}_1\cup\mathcal{L}_2)$ looks essentially like (annulus)$\times S^1$; if we choose the standard branched surface B carrying $\mathcal{L}_1\cup\mathcal{L}_2$, then M_0=$M\backslash$int(N(B)) has the structure of the stutured manifold (annulus)$\times S^1$=(torus)$\times I$, with two parallel sutures on each boundary component. By foliating M_0 by annuli, whose boundaries are not parallel to either of the sutures, we can, as in [Ga1], spin the leaves in M_0 along the annuli between the sutures to complete ($\mathcal{L}_1\cup\mathcal{L}_2$) to a foliation (Figure 5b).

Figure 5

The key fact in the proof of Theorem D is that no foliation of M which contains a (vertical) sublamination like one of the $\mathcal{L}_i$ can admit a transverse $C^{(2)}$ structure. This is because for every (annular) leaf of the sublamination, the foliation meets the normal fence over its core γ in one of the patterns of Figure 6; there are closed loops limiting on γ on one or both sides. This implies each leaf of the sublamination has trivial linear holonomy, if the foliation has class $C^{(2)}$. Proposition 4 says that this is impossible, however, since the sublamination does not form an open set in M.

Figure 6

If a $C^{(2)}$ foliation $\mathcal{F}$ of M has no compact leaves, then Proposition 1 and its extension imply that we can make $\mathcal{F}$ transverse to the splitting torus T, so that the induced foliations on the Seifert-fibered pieces M_i are essential. Each therefore contains a vertical or horizontal sublamination, by proposition 3. If a vertical sublamination misses T, then it comes from a 1-dimensional lamination in the interior of the base surface. It must therefore contain either a closed loop (giving a torus leaf of $\mathcal{F}$) or a lamination like the one above, so our original foliation cannot be made $C^{(2)}$. So any vertical sublamination must meet T, so the slope of the ∂-foliation is ∞, on one side. Since M_i does not contain a horizontal annulus, Proposition 2 implies that $\mathcal{F}$ cannot induce a Reeb foliated annulus on T, because it would have to be vertical when viewed from both sides, so M would be Seifert-fibered. Proposition 3c and the observation above imply that if the foliation on one of the M_i meets ∂M_i in a foliation of slope other than ∞, then the foliation is horizontal. Proposition 8 then implies that the induced slope is in [0,1). Put together, any $C^{(2)}$ foliation with no compact leaves in M can be made transverse to T; the induced foliations on M_i meet ∂M_i in slopes lying in $[0,1)\cup\{\infty\}$.

So to build the examples required for Theorem D, we need to find gluing maps A=(a,b;c,d) for which $A([0,1)\cup\{\infty\})\cap([0,1)\cup\{\infty\})=\emptyset$. As with the proof of Theorem B, this is easily arranged. Since $A(-d/c)=\infty$, $A(\infty)=a/c$, $A(0)=b/d$, and $A(1)=(a+b)/(c+d)$, after choosing $\det(A)=ad-bc=-1$ (for convenience, so that $A([0,1)$ will be $[A(1),A(0))$), we need

$-d/c\notin[0,1)$, $a/c\notin[0,1)$, and either $b/d < 0$ or $(a+b)/(c+d) > 1$.

One easy way to do this is to choose $c>0$, $d>0$, $a<0$, and $b<0$. for example,

A=(-1,-n;k,1+nk) with n,k$\geq$1, or

A=(-2,-(2n+1);2k+1,1+k+n+2nk) with n,k$\geq$0, work.

Because in each list the matrices have different traces, they are not conjugate, and so the glued manifolds are distinct.

Theorem E, on the other hand, follows immediately from the following theorem of Barbot:

Theorem [Ba]. *If M is as above, then M admits an Anosov flow if and only if the gluing map A is of the form $A=(-(kn+1),k;-n(kn+2),kn+1)$ where $n=1$ or 2.*

Every other possible gluing contains the foliation with no compact leaves that we built above, but does not admit any Anosov flows.

§5
Surgery on the (−2,3,7) pretzel knot

The $(-2,3,7)$ pretzel knot, also known as the Fintushel-Stern knot, is one of the most well-studied knots in the 3-sphere, second perhaps only to the Figure-8 knot. Its exterior $X(K) = S^3 \backslash \mathrm{int}(N(K))$ fibers over the circle, with pseudo-Anosov monodromy, and is therefore hyperbolic. By Thurston [Th], all but finitely-many Dehn fillings along K are hyperbolic. Bleiler and Hodgson [BH] have determined all of the surgeries along K which have finite fundamental group. With respect to the standard meridian/logitude coordinates on $\partial X(K)$, they are ∞, 17, 18, and 19. In addition, it has long been known [HO] that the manifold M obtained by 37/2 surgery on K contains an incompressible torus. This was, in fact, the first non-integral surgery on a hyperbolic knot (whose only closed incompressible surface in $X(K)$ is $\partial X(K)$) which was shown to contain an incompressible surface. Eudave-Muñoz [Eu1] has since shown that M is a graph manifold, obtained by gluing a left-handed and a right-handed trefoil knot exterior X_L and X_R together along their boundaries. Since trefoil knot exteriors are Seifert-fibered, with base a 2-disk and two multiple fibers, M can be analyzed as in our proof of Theorem B.

The gluing map A from ∂X_L to ∂X_R is most easily described in terms of the standard meridian/longitude coordinates for ∂X_L and ∂X_R. By Eudave-Muñoz [Eu2], A glues the meridian μ_L of ∂X_L to the circle fiber of the (induced) fibering of ∂X_R, and glues the circle fiber of ∂X_L to the meridian μ_R of ∂X_R. The fiber in ∂X_L is represented by $-6\mu_L+\lambda_L$ in the standard coordinates (where λ_L is the longitude in ∂X_L), while the fiber in ∂X_R is represented by $6\mu_R+\lambda_R$. This is perhaps most easily seen by comparing the boundary of a Seifert surface in X_L, say, to the boundary of the obvious Möbius band in X_L, which is the circle fiber in ∂X_L (since the Möbius band cuts X_L into a solid torus); see Figure 7.

Therefore, the gluing map A sends μ_L to $6\mu_R+\lambda_R$, and sends $-6\mu_L+\lambda_L$ to μ_R. This means that λ_L is sent to $37\mu_R+6\lambda_R$. In other words, with respect to the standard meridian/longitude coordinates on ∂X_L and ∂X_R, A is the matrix (6,37;1,6).

Figure 7

Given an essential lamination $\mathcal{L}$ in M, we can, as before, isotope it so that $\mathcal{L}$ is transverse to the splitting torus, and $\mathcal{L} \cap X_L$ and $\mathcal{L} \cap X_R$ are essential in X_L and X_R. In each piece it is therefore, by Proposition 3d, either horizontal or contains a vertical sublamination. The slopes realized by horizontal essential laminations in X_L and X_R are [Na1], in the standard meridian/longitude coordinates, $[-1,\infty)$ for X_L and $(-\infty,1]$ for X_R. But it is easy to see that the associated fractional linear transformation $\mathrm{A}(x)=(6x+37)/(x+6)$ sends $[-1,\infty)$ to $(6,31/5]$, since $[-1,\infty)$ does not contain the vertical asymptote -6 of $\mathrm{A}(x)$, and $\det(\mathrm{A})=-1$, so $\mathrm{A}(x)$ is decreasing on $[-1,\infty)$. Therefore the image is disjoint from $(-\infty,1]$, so two horizontal laminations cannot be glued together to form an (essential) lamination in M.

Finally, a lamination cannot be built from laminations containing a vertical sublamination in either piece, since a vertical lamination must consist, by Proposition 3d, either of a boundary parallel torus (our desired conclusion) or a collection of annuli separating the two multiple fibers of the fibering of the knot exterior (and perhaps a Möbius band containing the multiplicity 2 fiber). Such an annulus separates X_L (say) into two solid tori, which the lamination meets in horizontal leaves. It therefore meets $\partial X_L = T$ in vertical loops (from the vertical sublamination) with Reeb type leaves in between. But since the fiber in ∂X_L is glued to the meridian in ∂X_R (and vice versa), this means that $\mathcal{L}$ meets $\partial X_R = T$ (say) in meridian loops with Reeb leaves in between. But this contradicts Proposition 3d.

Therefore, no essential laminations in X_L and X_R can be glued together to give a lamination in M, unless one contains a parallel copy of the splitting torus. In other words, every essential lamination in M contains the splitting torus as a leaf. Since this torus intersects the image of K (it can, in fact [Eu1], be made to intersect it in exactly two points), we find in particular that every essential lamination in M intersects the image of K. Therefore, there is no essential lamination in $S^3\backslash K$ (i.e., no essential lamination $\mathcal{L}$ in $S^3\backslash\mathrm{int}\ \mathrm{N(K)} = X(K)$ with $\mathcal{L}\cap\partial\mathrm{X(K)} = \emptyset$) which remains essential after 37/2 surgery along K.

It was shown by Christy [Ch] that the essential lamination in X(K), obtained by taking the suspension of the stable 1-dimensional lamination for the monodromy of the fibering of X(K), has degeneracy slope (see [Ga3]) equal to 18/1. This lamination therefore remains essential under every Dehn surgery along K, except those with surgery coefficient of the form (∞ or 18 or) $18 \pm 1/\mathrm{n}$, for $\mathrm{n}\geq 1$. Among these surgery coefficients, the only ones which give manifolds which are known not to contain essential laminations are ∞,17,18, and 19, since these manifolds all have finite fundamental group. It has been a long-standing open problem (as long-standing as anything in a field that is only ten years old can be, anyhow) to show that all of the other surgeries yield laminar manifolds (or to prove that one of them doesn't - this would probably yield the first example of a hyperbolic, non-laminar, 3-manifold). Several people have attempted to do this by finding an essential lamination in $S^3\backslash K$ with degeneracy slope 1/0, since the lamination would then remain essential under every non-integral surgery. The above result, however shows that this is impossible; it could not remain essential under 37/2 surgery. In fact, the result also shows that every essential lamination in $S^3\backslash K$ must have degeneracy slope a curve with intersection number 0 or 1 with the slopes ∞, 17, 18, 19, and 37/2 (since otherwise the lamination would remain essential in one of the resulting manifolds). The only slope for which this is true is 18/1, so every essential lamination in $S^3\backslash K$ has degeneracy slope 18/1.

These observations leave open the possiblity, however, of finding essential laminations in X(K), which meet ∂X(K) in curves with these missing slopes, and which remain essential after Dehn filling and capping off the boundary curves with disks. This, for example, is how the laminations of [Na1],[Ro1] are constructed. Any such construction must be somewhat subtle, however, since any lamination constructed for slope 37/2 (and no other, since by [HO] every other missing slope gives a non-Haken manifold) must contain a compact leaf.

§6
The future

This paper demonstrates that the set of manifolds admitting the various topologically useful classes of foliations are all distinct. This suggests that a workable necessary and sufficient condition for the existence of these classes of foliations will be difficult, if not impossible, to find. This contrasts with the case of embedded incompressible surfaces, for example, which admits a fairly succinct (although perhaps not practical) existence criterion; a 3-manifold M contains an incompressible surface if and only if the fundamental group of M is a free product with amalgamation or HNN extension over a surface group [Fu],[FG]. We should also point out that the work on essential laminations and foliations in closed Seifert-fibered spaces ([Br1],[Cl],[EHN],[JN],[Na1]), which we have relied on throughout this work, has already demonstrated that while among non-Haken Seifert-fibered spaces, the manifolds admitting these different classes of foliations are of course all identical - any foliation with no Reeb components has all leaves π_1-inject, so has no compact leaves - the 'dividing line' between those which do have essential foliations and those which don't [JN],[Na1] is extremely complicated. One good open question, in fact, is to find an explanation (in terms of the fundamental group, perhaps) for this 'dividing line'.

For a hyperbolic 3-manifold M, however, many of the distinctions we have explored here disappear. A closed hyperbolic 3-manifold contains no incompressible tori, so a foliation without Reeb components has no torus leaves, and so is automatically taut. Therefore, only a few of these distinctions survive.

Question. *Does every hyperbolic 3-manifold admit a taut foliation?*

Question. *Does every tautly-foliated hyperbolic 3-maniflold admit a foliation with no compact leaves?*

Question. *Does every tautly-foliated hyperbolic 3-maniflold admit an $\mathbb{R}$-covered foliation?*

A foliation is $\mathbb{R}$-covered if the space of leaves of the foliation, after being lifted to the universal cover of M, is the real line $\mathbb{R}$. Tautness is a necessary condition for a foliation to be $\mathbb{R}$-covered. Among non-Haken Seifert-fibered spaces, every taut foliation is $\mathbb{R}$-covered [Br4].

Question. *Does every tautly-foliated hyperbolic 3-maniflold admit a non-$\mathbb{R}$-covered foliation?*

We note that the answers to these last two questions are 'No', in general; there are, again, counterexamples among graph manifolds, see [Br3].

The answers to these questions remain out of the reach of present technology - our current understanding of the structure of taut foliations of hyperbolic 3-manifolds is rather limited. The best results to date are those of Fenley [Fe1,Fe2] who has some interesting results on the structure of stable foliations of Anosov flows on hyperbolic manifolds, as well as on the limit sets of leaves of foliations in hyperbolic 3-manifolds.

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