Essential laminations and deformations of homotopy equivalences, II: the structure of pullbacks

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§0
Introduction

It is a long-standing conjecture in 3-manifold theory that every homotopy equivalence $f{:}M{\to}N$ between closed, irreducible 3-manifolds M and N, with $|\pi_1(M)|$ $(=|\pi_1(N)|)=\infty$, is homotopic to a homeomorphism. In [Wa], Waldhausen proved that this conjecture was true, if we assume that N contains a 2-sided incompressible surface S. The proof consists of first homotoping the map f so that the pullback surface $f^{-1}(S)$ is an incompressible surface, and then splitting both manifolds open along these surfaces, and proceeding by induction on a hierarchy of N - a collection of incompressible surfaces in the successively split-open manifolds which end up splitting N into a 3-ball. The base case is the Alexander trick.

In [Br1], we began a program to extend this theorem to the case that N is *laminar*, i.e., N contains an essential lamination [G-O]. We showed that, given a homotopy equivalence $f{:}M{\to}N$ between non-Haken, (irreducible) 3-manifolds and a transversely-orientable essential lamination $\mathcal{L}{\subseteq}N$, if the pullback lamination $f^{-1}(\mathcal{L}){\subseteq}M$ is essential then we could homotope the map f to a homeomorphism. In this paper we study the 'other half' of Waldhausen's proof - when can one deform a homotopy equivalence to give an essential pullback - by trying to understand what structure the pullback of an essential lamination has, in general, under a homotopy equivalence. We work, as in [Br1], under the hypothesis that M and N are non-Haken; otherwise, all that we shall use about the map f is that it has degree 1 (in particular, non-zero degree).

It has been conjectured that the method employed by Waldhausen - surgeries of the pullback achieved as homotopies of the map f - will be successful (with some modification) in creating an essential pullback in the present context. We begin this paper by showing that if the homotopy-through-surgery process will succeed in making the pullback essential, then, at the very beginning, the pullback must have had a fairly restrictive structure - each leaf of the pullback must map to its corresponding leaf in $\mathcal{L}$ with degree 0 or 1. In the remainder of the paper we attempt to show that leaves of a pullback do in fact have this property. We must admit at the outset that we do not succeed in proving this - but we will see that the

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pullback has many of the same 'tightly-wrappedness' properties that an essential lamination in a non-Haken manifold has [Br2], which 'almost' implies that all leaves have degree 0 or 1.

§1
The homotopy through surgery process

> **Note:** we assume, as in [B1], that the essential lamination $\mathcal{L}$ has every leaf dense in $\mathcal{L}$, i.e., for every leaf L of $\mathcal{L}$, $\overline{L} = \mathcal{L}$. This necessary hypothesis can be easily arranged, by throwing away all but a minimal sublamination of any essential lamination $\mathcal{L}$ that we start with. We also assume that $\mathcal{L}$ contains no open sets in M, i.e., that transversely $\mathcal{L}$ looks like a nowhere-dense closed subset of an interval (typically, a Cantor set). This can be arranged, if necessary, by splitting $\mathcal{L}$ open along a finite number of leaves.

Given a homotopy equivalence $f{:}M{\to}N$, where M and N are closed irreducible 3-manifolds, and an essential lamination $\mathcal{L}{\subseteq}N$, we say that f is *transverse* to $\mathcal{L}$ if, for every point $x{\in}f^{-1}(\mathcal{L})$, $T_{f(x)}(\mathcal{L}) + f_*(T_x(M) = T_{f(x)}(N)$. This is most easily achieved by making f transverse to a *branched surface* B which carries $\mathcal{L}$; then by imagining that $\mathcal{L}$ lies very close to B, f is immediately transverse to $\mathcal{L}$. This method of achieving transversality is the one that we tend to think of as 'standard'. However, we will occasionally make use of the more general point of view, especially when doing homotopies through surgery. Since for the most part we will be doing surgeries in order to eventually lead to a contradiction, these 'excursions' will from one point of view be unnecessary (hence our adoption of the more restrictive notion as 'standard').

Given a map $f{:}M{\to}N$ transverse to the lamination $\mathcal{L}{\subseteq}N$, the pullback $\mathcal{L}' = f^{-1}(\mathcal{L})$ is a lamination. If the map f is transverse to the branched surface B carrying $\mathcal{L}$, then the pullback lamination $\mathcal{L}'$ is carried by the pullback branched surface $B' = f^{-1}(\text{B})$.

The main construction in our proofs will be that of <u>surgering</u> the pullback lamination $\mathcal{L}'$. This begins with a situation as in Figure 1a - a loop in leaf of $\mathcal{L}'$, and a disk D, which is embedded in M, but not necessarily embedded in $M|\mathcal{L}'$, and which compresses it. We assume that D is transverse to $\mathcal{L}'$, so $D{\cap}\mathcal{L}'$ is a 1-dimensional lamination in M. Note that since $\mathcal{L}$ is essential, $D{\cap}\mathcal{L}'$ consists of (a finite number of parallel families of) closed loops - no loop of $D{\cap}\mathcal{L}'$ can have any holonomy around them. This is because they are null-homotopic in M, so their images under f are null-homotopic in N, so (by the essentiality of $\mathcal{L}$) their images under f are null-homotopic in the leaves containing them, hence have no holonomy around them. So the original loops have no holonomy - holonomy is preserved under transverse maps. There can also be no non-compact leaves which <u>don't</u> limit on compact ones, since this would imply (by pushing forward under f) the existence of an end-compression for $\mathcal{L}$.

We can then create a new lamination in M, by surgering $\mathcal{L}'$ along this disk, by cutting $\mathcal{L}'$ open along D, and sewing in a collection of disks to the resulting circles, working from innermost parallel family out (see Figure 1b). As with 'ordinary' surgery of a compact surface (see [He]), this operation can be achieved by a homotopy of f; the new lamination is the pullback of a map homotopic to (and which we will still call) f.

Figure 1

In many cases (e.g., with a Reeb component (Figure 2)), this surgery will create sphere leaves, or (necessarily inessential) leaves contained inside of embedded 3-balls; but since we assume that both M and N are irreducible (so are both K(π, 1)'s), these spheres can be removed via a homotopy of f, as well, in the usual way(see [Wa].

Figure 2

§2
The degree of a leaf should be 0 or 1

In this section we prove:

Proposition 1. *If a finite number of surgeries like those described above can make the pullback lamination $\mathcal{L}'$ essential (modulo leaves lying in 3-balls), then every leaf of the pullback lamination maps to its corresponding leaf of $\mathcal{L}$ with degree either 0 or 1.*

This follows fairly directly from the following results:

Lemma 2. *Given $f{:}M{\to}N$, $\mathcal{L}$, and $\mathcal{L}' = f^{-1}(\mathcal{L})$ as above, let $\mathcal{L}'_0$ be the result of surgering $\mathcal{L}'$, achieved as a homotopy of the map f to the map g. Then for any leaf L' of $\mathcal{L}'$, mapping under f to the leaf L of $\mathcal{L}$, if L'_i, i=1,2,... are the leaves of $\mathcal{L}'_0$ that L' was surgered into, then*

$$d(f|_{L'}) = \Sigma_{i=1}^{\infty} d(g|_{L'_i}),$$

where d() refers to the degree of the map, thought of as a map into L.

Lemma 3. *With the same initial conditions, if a leaf L' of $\mathcal{L}'$ is entirely contained in the 3-ball, then $d(f|_{L'}) = 0$.*

Proof of Proposition 1: From [B1, Proposition 7], we know that when $\mathcal{L}'$ is essential, every leaf of $\mathcal{L}'$ maps to its corresponding leaf of $\mathcal{L}$ with degree 1. By our hypothesis, after a finite number of surgeries, for each leaf L of $\mathcal{L}$, $f^{-1}(L)$ consists of a single leaf mapping with degree 1, and a collection of leaves in 3-balls, which by Lemma 3 map with degree 0; in particular, the degrees of leaves are all non-negative. We can therefore inductively conclude, by undoing each surgery and appealing to Lemma 1, that the degree of each leaf at each stage is the sum of the degrees of leaves, all of which are non-negative. Consequently, all leaves of $f^{-1}(\mathcal{L})$ map with non-negative degree. But for each leaf L of $\mathcal{L}$, the degrees of each leaf of $f^{-1}(L)$ must add up to the degree of $f{:}M{\to}N$, which is 1, so we can conclude that one leaf of $f^{-1}(L)$ maps with degree 1, and all others map with degree 0.

Proof of Lemma 2:

The general case is not much different, requiring just a bit more attention to detail. By cutting up the family of loops around the innermost one into a finite number of 'thin' pieces (using the fact that transversely they are nowhere dense), we can assume by deforming the map slightly that the family of loops map in 'vertically', i.e., the image of the annular region between innermost and outermost loop maps to the normal fence over the image of the innermost loop. If we then choose a null-homotopy of f of each loop to achieve our surgery, this null-homotopy will lift to the nearby leaves. By cutting our family of loops into finitely-many even thinner families, we can then assume that every null homotopy for our loops is the lift of the null homotopy for the innermost one. Now we replace, as in [He], a thin annular region in each leaf with two copies of these null-homotopies; this can be achieved by a homotopy of f. In this more genereal case, this could separate a leaf into infinitely-many leaves, but because the sewn-in disks all cancell out in any degree calculation, the total degree of the surgered leaf remains the same. A finite number of applications of this completes the proof.

Proof of Lemma 3: Since $\overline{L'}$ is contained in the ball B^3, the inclusion-induced map $\pi_1(L') \to \pi_1(M)$ is the zero map, and consequently (if $f(L') \subseteq L \subseteq \mathcal{L}$), the map $f_* \pi_1(L') \to \pi_1(L)$ must be the zero map (since composing with the inclusion-induced <u>injective</u> map $\pi_1(L) \to \pi_1(M)$ gives the zero map).

If L is not simply-connected, then $|\pi_1(L)| = \infty$. By the lifting criterion, there is a map $\tilde{f}: L' \to \widetilde{L}$ (where $\widetilde{L}$ is the universal cover of L), so that

$$\begin{tikzcd} L' \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \widetilde{L} \arrow[d, "p"] \\ & L \end{tikzcd}$$

commutes. But now choose a regular value x for $f: f^{-1}(L) \to L$ (so x is also a regular value for $f: L' \to L$). then $f^{-1}(x) = \tilde{f}^{-1}(p^{-1}(x) = \tilde{f}^{-1}(\cup x_i) = \cup \tilde{f}^{-1}(x_i)$, where $\{i\}$is an infinite index set. Since f is transverse to $\mathcal{L}$, however, x is also a regular value for $f: M \to N$, so $f^{-1}(x)$ consists of finitely-many points. Consequently, most of the $\tilde{f}^{-1}(x_i)$ must be empty, so $d(\tilde{f})=0$. Therefore the signed sum over each $\tilde{f}^{-1}(x_i)$ must also be 0, so the signed sum over $f^{-1}(x)$ (for $f: L' \to L$) is a sum of zeros, so $d(f: L' \to L) = 0$.

If L <u>is</u> simply-connected, we will argue by contradiction. If L' maps with non-zero degree, then $f(L') = L$, so $f(\overline{L'})$ is a (compact, hence) closed set in N containing L, hence contains $\overline{L} = \mathcal{L}$. But $\mathcal{L}$ contains a non-simply connected leaf L_0 (else N is the 3-torus [Ga1], hence Haken), so $\overline{L'}$ contains leaves (in the 3-ball), namely $f^{-1}(L_0) \cap \overline{L'}$, which map to a non-simply-connected leaf, hence all map with degree 0.

§3
The structure of pullbacks, I:
degree-zero vs. non-zero degree leaves

In this section we begin to establish some of the properties that a pullback lamination must always have. For this section, we assume only that the lamination $\mathcal{L}$ has every leaf dense (and, tacitly, that the map f has non-zero degree, so that <u>some</u> leaf of $f^{-1}(\mathcal{L}) = \mathcal{L}'$ must map with non-zero degree). In many respects, the arguments mimic those of [B1, Section 2].

Proposition 4. *If $\mathcal{L}_0 \subseteq f^{-1}(\mathcal{L})$ is a sublamination, and z is a regular value of f in $\mathcal{L}$, with I a (short open) arc though z transverse to $\mathcal{L}$, then for every $x \in I \cap \mathcal{L}$, and every $\epsilon > 0$, there exists $y \in I \cap \mathcal{L}$ with $d(x,y) < \epsilon$ and an open $\mathcal{U} \subseteq I$ containing y such that (for $f^{-1}(I) = I_1 \cup I_2 \cup \ldots \cup I_n$) for each $j = 1, \ldots, n$, setting $f^{-1}(\mathcal{U}) \cap I_j = \mathcal{U}_j$, either $\mathcal{U}_j \cap f^{-1}(\mathcal{L}) \subseteq \mathcal{L}_0$ or $\mathcal{U}_j \cap \mathcal{L}_0 = \emptyset$.*

Proof: $f^{-1}(I) = I_1 \cup \cdots \cup I_n$, each mapping homeomorphically to I under f. Now given $x \in I$, and $\epsilon > 0$, choose a $y \in I$ with $d(x,y) < \epsilon$ such that $f^{-1}(y) \cap \mathcal{L}_0$ consists of the fewest number of points; call them $\{y_1, \ldots, y_k\}$. Note that, for $i \geq k+1$, since $y_i \notin \mathcal{L}_0$, and $\mathcal{L}_0$ is closed, all of the points in I_i sufficiently close to y_i are not in $\mathcal{L}_0$; for some open $\mathcal{U}_i$ in I_i containing y_i, $\mathcal{U}_i \cap \mathcal{L}_0 = \emptyset$ (sse Figure 3).

Figure 3

Claim: for $i \leq i \leq k$, all points of $f^{-1}(\mathcal{L})$ sufficiently close to the y_i are in $\mathcal{L}_0$.

For if not, then (passing to a subsequence and renumbering as necessary) there is a sequence of points $z_1, z_2, \ldots$ in I_i, converging to y_i, such that $z_j \notin \mathcal{L}_0$ for every j. Renumber so that $i = k$. Then if we set $x_j = f(z_j) \in I$, then the x_j converge to y, and so for large enough j, the x_j's are within ϵ of x. But for large enough j, $f^{-1}(x_j) \cap I_i$ is a point in $\mathcal{U}_i$, for $k+1 \leq i \leq n$, so is not in $\mathcal{L}_0$, while $f^{-1}(x_j) \cap I_k = z_k$, so is also not in $\mathcal{L}_0$. So $f^{-1}(x_j)$ contains fewer points of $\mathcal{L}_0$ than $f^{-1}(y)$ does, a contradiction.

Therefore, for $1 \leq i \leq k$, there is an open interval $\mathcal{U}_i$ of I_i about y_i such that $\mathcal{U}_i \cap f^{-1}(\mathcal{L}) \subseteq \mathcal{L}_0$. Now set $\mathcal{U} = f(\mathcal{U}_1) \cap \cdots \cap \mathcal{U}_n$. ∎

Corollary 5. *If L' is a leaf of $f^{-1}(\mathcal{L})$, and $\mathcal{L}_0$ is a proper sublamination of $\overline{L'}$, then $f(\mathcal{L}_0)$ is nowhere dense in $\mathcal{L}$ (i.e., $\mathcal{L} \setminus f(\mathcal{L}_0)$ is open and dense in $\mathcal{L}$). In particular, every leaf of $\mathcal{L}_0$ maps to $\mathcal{L}$ with degree zero.*

Proof: By the proof of the proposition, for every regular value x of f in $\mathcal{L}$, and $\epsilon > 0$, there is a regular value y of f in $\mathcal{L}$, within ϵ of x along a transverse arc I, and an open interval $\mathcal{U}$ in I about y such that $\mathcal{U}_j \cap f^{-1}(\mathcal{L}) \subseteq \mathcal{L}_0$ or $\mathcal{U}_j \cap \mathcal{L}_0 = \emptyset$ But our hypothesis implies that for each j it must be that $\mathcal{U}_j \cap \mathcal{L}_0 = \emptyset$; for otherwise given any point $z \in \mathcal{U}_j \cap f^{-1}(\mathcal{L}) \subseteq \mathcal{L}_0$, L', which is not in $\mathcal{L}_0$, passes arbitrarily close to z, and hence intersects the open arc component $\mathcal{U}_j$ that contains z, a contradiction. Therefore $\mathcal{U} \cap f(\mathcal{L}_0) = \emptyset$; in particular, $y \notin f(\mathcal{L}_0)$. So $A = \mathcal{L} \setminus f(\mathcal{L}_0)$ is an open subset (since $f(\mathcal{L}_0)$ is the continuous image of a compact set) of $\mathcal{L}$, and any (regular) point of $\mathcal{L}$ has a point of A arbitrarily close (in the transverse direction) to it (since x was arbitrary). So A is dense in $\mathcal{L}$ (since regular values are dense in $\mathcal{L}$). In particular, A contains a regular value x for f in $\mathcal{L}$. For any leaf L_0' of $\mathcal{L}_0$, mapping under f to the leaf L_0 of $\mathcal{L}$, since L_0 passes arbitrarily close to x (all leaves of $\mathcal{L}$ are dense in $\mathcal{L}$), L_0 contains a nearby regular value y in A (since A is open). Then $f^{-1}(y) \cap \mathcal{L}_0 = \emptyset$, so $f^{-1}(y) \cap L_0' = \emptyset$, so L_0' maps to L_0 with degree zero. ∎

Corollary 6. *If L_0', L' are leaves of $\mathcal{L}'$ mapping with non-zero degree, and $L_0' \subseteq \overline{L'}$, then $\overline{L_0'} = \overline{L'}$.* ∎

Let $\mathcal{H}$ denote the leaves of $\mathcal{L}$ having no holonomy; by [EMT], this is a dense subset of $\mathcal{L}$. Let $\mathcal{H}' = f^{-1}(\mathcal{H})$; this is a (saturated) subset of the set of leaves of

$f^{-1}(\mathcal{L}) = \mathcal{L}'$ having no holonomy, since a loop in $\mathcal{H}'$ with non-trivial holonomy would be carried under f to one in $\mathcal{H}$ with non-trivial holonomy, since transverse pictures are preserved under f. $\mathcal{H}'$ is also dense in $\mathcal{L}'$, because transversely it looks the same as $\mathcal{H}$ in $\mathcal{L}$ (again, since transverse pictures are preserved under f).

Proposition 7. *If $L'\subseteq\mathcal{L}'$ maps to $L\subseteq\mathcal{L}$ under f, with non-zero degree, then ($f(\overline{L'})=\mathcal{L}$, so) $\mathcal{H}_{L'}=\overline{L'}\cap\mathcal{H}'\neq\emptyset$, every leaf of $\mathcal{H}_{L'}$ maps to its corresponding leaf either with degree 0 or with the <u>same</u> (non-zero) degree $d = d(\overline{L'})$, and every leaf of $\overline{L'}$ maps to its corresponding leaf with degree a <u>non</u>-<u>negative</u> multiple of d.*

Proof: Pick a regular value x for f in a leaf of $\mathcal{H}$, satisfying the conclusions of Proposition 4. That is, pick a regular value in a leaf of $\mathcal{L}$ satisfying the conditions, and then (since $\mathcal{H}$ is dense in $\mathcal{L}$) pick a nearby point in $\mathcal{H}$. Let $f^{-1}(x)\cap\overline{L'}$ $= \{x_1,\dots,x_k\}\subseteq\mathcal{H}_{L'}$. One of the leaves that these points lie in must map to its corresponding ($\mathcal{H}$-) leaf with non-zero degree; otherwise, following the argument of [B1, Lemma 5], we can join these points together by arcs in their leaves, as the vertices of a tree, such that for each component of the tree the sum of the signs of its vertices is zero (i.e., the degree of the leaf containing it). Then by lifting these trees to nearby leaves (see Figure 4), since the resulting endpoints are all in the inverse image under f of the same point (because we are lifting from a leaf with no holonomy), we see that the degree of every nearby leaf (including L') is a sum of 0's, hence 0, a contradiction. So some $\mathcal{H}_{L'}$-leaf L'_0 has non-zero degree, so by Corollary 5, $\overline{L'_0} = \overline{L'}$. But now we can easily see, first, that every leaf of $\mathcal{H}_{L'}$ the <u>same</u> degree; this is because each is dense in $\overline{L'}$ (by Corollary 6), and so they pass arbitrarily close to one another. The argument of [B1, Lemma 5], applied verbatim, then gives the result. The final statement then follows by taking all of the points of $\{x_1,\dots,x_k\}\subseteq\mathcal{H}_{L'}$ and stringing them together in their leaves (so the resulting trees each have vertex-sum either d or 0), and then lifting them to the leaves containing any other inverse image of a point close to x. The leaves have degree the sum of the degrees of the point inverses that lie in them, but since our trees lift to trees in these leaves (so any leaf containing one vertex contains them all), this sum is a sum of d's and 0's, hence is a non-negative multiple of d. ∎

Figure 4

Proposition 8. *If L' maps with non-zero degree, then the set $\mathcal{N}_{L'}$ of leaves in $\overline{L'}$ which map with non-zero degree is open in $f^{-1}(\mathcal{L}) = \mathcal{L}'$.*

Proof: Let x be a regular value chosen as in Proposition 7, and let L'_0 be a $\mathcal{H}_{L'}$-leaf mapping with non-zero degree d. Let I be an arc, transverse to $\mathcal{L}$, passing through x, and I_1 the arc in the inverse image of I passing through a point $x_1$of $f^{-1}(x)$ in L'_0. Assume that x=y has been chosen so that the number distinct leaves of $\mathcal{H}_{L'}$ meeting $f^{-1}(y)$ is minimal among the points y lying in $I\cap\mathcal{H}$ (otherwise, move to a different point!).

Claim: All of the leaves of $\mathcal{H}_{L'}$ meeting $f^{-1}(x)$ map with non-zero degree.

Proof of claim: As before, join together the points of $f^{-1}(x)\cap\overline{L'}$ contained in the same leaves, as the vertices of a tree; this tree lifts to nearby leaves, and the vertices of the lifted trees all lie in the inverse image of a single point lying near x.

If L_1' is a $\mathcal{H}_{L'}$-leaf meeting $f^{-1}(x)$ and mapping with degree 0, then, since $\overline{L_0'}=\overline{L'}$, L_0' limits on L_1'. But then when we lift the trees to L_0'(which corresponds to a point z near x), the vertices of the component contained in L_1', since their degrees add up to 0, do not contain all of the points of $f^{-1}(z)\cap L_0'$. So more than one component of the lifted tree lives in the same leaf, so the number of leaves of $\mathcal{H}_{L'}$ intersecting $f^{-1}(z)$ is smaller, contradicting our choice of x.

This in turn allows us to conclude that <u>every</u> leaf sufficiently close to L_0' maps with non-zero degree. All of the leaves of $\overline{L'}$ meeting $f^{-1}(x)$ map to the leaf containing x with the same degree $d\neq 0$. Building our by now familiar trees joining the points of $f^{-1}(x)$ together, and lifting these trees to any nearby point-inverse, we see that the degrees of these nearby leaves (in $\overline{L'}$) are positive multiples (each contains at least <u>one</u> such tree) of d, and hence are all non-zero. Since any non-zero degree leaf L_1' in $\overline{L'}$ limits on L_0', L_1' is among these nearby leaves, so all of the leaves <u>around</u> it have non-zero degree (and are contained in $\overline{L'}$). So the non-zero degree leaves of $\overline{L'}$ form an open set in $\mathcal{L}'$. ∎

Corollary 9. *If L' maps to L with non-zero degree, then $\mathcal{Z}_{L'}$, the set of leaves of $\overline{L'}$ mapping to $\mathcal{L}$ with degree 0, is a closed set, forming a proper sublamination of $\overline{L'}$ (so $f(\mathcal{Z}_{L'})\subseteq\mathcal{L}$ is nowhere dense in $\mathcal{L}$).*

Proof: By Proposition 8, $\mathcal{Z}_{L'}=\overline{L'}\setminus\mathcal{N}_{L'}$ is a (relatively) closed set in $\overline{L'}$, hence is closed. So it is a sublamination of $\overline{L'}$. Because $L'\notin\mathcal{Z}_{L'}$, $\mathcal{Z}_{L'}$is a <u>proper</u> sublamination of $\overline{L'}$. ∎

Corollary 10. *A leaf L_0' of $\mathcal{L}'$ mapping with degree zero cannot limit on a leaf L' mapping with non-zero degree (i.e., $\overline{L_0'}$ consists of degree-zero leaves).*

Proof: Suppose it could; $L'\subseteq\overline{L_0'}$. We cannot have $L_0'\subseteq\overline{L'}$, since then $L_0'\subseteq\mathcal{Z}_{L'}$, which is, by Corollary 9, a (<u>closed</u>) sublamination consisting of degree-zero leaves, yet $L'\subseteq\overline{L_0'}\subseteq\mathcal{Z}_{L'}$, a contradiction. Consequently, $\overline{L'}$is a <u>proper</u> sublamination of $\overline{L_0'}$. But then Corollary 5 implies that every leaf of $\overline{L'}$ (e.g, L') maps with degree zero, a contradiction. ∎

Corollary 11. *The set $\mathcal{Z}$ of leaves of $\mathcal{L}'$ which map under f with degree zero is a closed set, hence forms a sublamination of $\mathcal{L}'$.* ∎

If L' is a leaf mapping with non-zero degree (to the leaf L), then we know that $f(\overline{L'})=\mathcal{L}$. But even more, since $\mathcal{N}_{L'}$ is open in $\mathcal{L}'$, if we pick a regular value x for f in L, some inverse image of x is in L', and since a small transverse arc I through x intersects every leaf of $\mathcal{L}$ (since every leaf is dense), and the component of the inverse image of I meeting L' intersects only leaves of $\mathcal{N}_{L'}$, this means that every leaf of $\mathcal{L}$ is the image of some (non-zero degree) leaf of $\mathcal{N}_{L'}$; i.e., $f(\mathcal{N}_{L'})=\mathcal{L}$.

Corollary 12. *For any two non-zero degree leaves L_1', L_2', either $\mathcal{N}_{L_1'}=\mathcal{N}_{L_2'}$ or $\mathcal{N}_{L_1'}\cap\mathcal{N}_{L_2'}=\emptyset$.*

Proof: Suppose $\mathcal{N}_{L_1'}\cap\mathcal{N}_{L_2'}\neq\emptyset$, so there is a leaf L' contained in both (hence mapping with non-zero degree). Since $L'\subseteq\mathcal{N}_{L_i'}\subseteq\overline{L_i'}$ for i=1,2, $\mathcal{N}_{L'}\subseteq\overline{L'}\subseteq\overline{L_i'}$, so $\mathcal{N}_{L'}\subseteq\mathcal{N}_{L_i'}$ for i=1,2. If either of these containments is <u>proper</u> (say, for i=1), then there is a (non-zero degree) leaf of $\mathcal{N}_{L'_1}$ not contained in $\mathcal{N}_{L'}$, hence not contained in $\overline{L'}$. So $\overline{L'}$ is a <u>proper</u> sublamination of $\overline{L'_1}$, so, by Corollary 5, all of its leaves (including L') maps with degree 0, a contradiction. Consequently, $\mathcal{N}_{L'_1}=\mathcal{N}_{L'}=\mathcal{N}_{L'_2}$. ■

Corollary 13. *The set $\mathcal{N}$ of leaves of $\mathcal{L}'$ mapping with non-zero degree can be written as a disjoint union of finitely-many of the sets $\mathcal{N}_{L'_1},\dots,\mathcal{N}_{L'_k}$.*

Proof: For each leaf L' mapping with non-zero degree, $f(\mathcal{N}_{L'})=\mathcal{L}$. If we pick a regular value x for f in a leaf L of $\mathcal{L}$, $f^{-1}(x)$ consists of finitely-many points, and since every set $\mathcal{N}_{L'}$ has a leaf mapping (with non-zero degree) onto L, each set $\mathcal{N}_{L'}$ contains at least one of the points of $f^{-1}(x)$. Consequently (since any two of the sets $\mathcal{N}_{L'}$ are either disjoint or identical) there are only finitely-many distinct sets of the form $\mathcal{N}_{L'}$. Since any leaf $L'\subseteq\mathcal{N}$ is contained in one of these sets (the set $\mathcal{N}_{L'}$, in fact!), the result follows.

§4
The structure of pullbacks, II: tightly-wrappedness

Now we add the hypothesis that our domain manifold, M, is irreducible and non-Haken. This allows us to mimic the arguments of [B2].

Proposition 14. *If L'_1 and L'_2 are leaves of $\mathcal{L}'$ mapping with non-zero degree, then $\overline{L'_1}\cap\overline{L'_2}\neq\emptyset$.*

Proof: If $\overline{L'_1}\cap\overline{L'_2}=\emptyset$, then the two laminations have disjoint (fibered) neighborhoods, so they can be separated by a finite collection F of disjoint compact surfaces (e.g., the boundaries of one of the fibered neighborhoods); $\overline{L'_1}$ and $\overline{L'_2}$ are contained in distinct components of $M\backslash F$. Because M is non-Haken, F cannot be incompressible, and, in fact, there must be a sequence of compressions of F (through surfaces $F=F_0, F_1, \dots, F_n$) so that F_n is a collection of disjoint 2-spheres; $F_n = S_1\cup\cdots\cup S_k$. Because M is irreducible, each S_i bounds a 3-ball B_i.

Lemma 15. *At most one component of $M\backslash F_n$ is <u>not</u> contained in one of the 3-balls B_i.*

Proof: This is a consequence of the fact that every surface in an irreducible, non-Haken 3-manifold M separates M. Every one of our 2-shperes S_i bounds a 3-ball B_i on only one side (otherwise $M=S^3$, which can't map, with non-zero degree, to a manifold having infinite fundamental group). This allows us to define a partial ordering on the S_i; $S_i<S_j$ if $B_i\subseteq B_j$. If we take the maximal elements under this ordering, then they form a collection F' of spheres bounding disjoint 3-balls. They also split M into one more component than the number of spheres, so exactly one component, M_0, <u>isn't</u> a 3-ball. M_0 is a component of $M\backslash F_n$, because any 2-sphere of F_n, in M_0, would not be maximal, so would be contained in one of the 3-ball components of $M\backslash F'$, and hence would <u>not</u> be in M_0. So every component of $M\backslash F_n$, except M_0, is contained in one of the maximal 3-balls. ■

The two sublaminations $\overline{L'_1}$ and $\overline{L'_2}$ will meet our compressing disks; otherwise, by the lemma, one or the other is contained in a ball, so all of its leaves map with degree zero, a contradiction. But we can make $\mathcal{L}'$ transverse to these disks; $\mathcal{L}'$ then meets the disks in a collection of circles and arcs. More importantly, $\overline{L_1'}$ and $\overline{L_2'}$ meet the disks in a collection of circles only, because an arc of interscetion would require one of them to intersect our surface F. If we first surger the $\overline{L_i'}$ along these circles, we can then surger F without intersecting the (surgered) $\overline{L_i'}$. So after surgery (whose effect we will denote by $s()$), $s(\overline{L_1'})$ and $s(\overline{L_2'})$ are disjoint from the 2-spheres, so, again, one of them ($s(\overline{L_1'})$, say) is contained in a ball. So Lemma 2 implies that every leaf of $s(\overline{L_1'})$ maps with degree zero. But because L_1' maps with non-zero degree, Lemma 3 and induction on the number of surgeries involved in s implies that some leaf of s($\overline{L_1'}$) maps with non-zero degree, a contradiction. So $\overline{L_1'}\cap\overline{L_2'} \neq \emptyset$. ∎

Proposition 16. *If L_1', ..., L_n' are leaves of $\mathcal{L}'$ mapping with non-zero degree, then $\overline{L_1'}\cap\cdots\cap\overline{L_n'} \neq \emptyset$.*

Proof: This follows by induction on n. n=2 is Proposition 15. If n is the smallest number where we have leaves with $\overline{L_1'}\cap\cdots\cap\overline{L_n'} = \emptyset$, then $\overline{L_1'}\cap\cdots\cap\overline{L_{n-1}'}$ $= \mathcal{L}_0' \neq \emptyset$, and $\mathcal{L}_0'\cap\overline{L_n'} = \emptyset$. But now if we apply the argument of Proposition 15, there is a sequence of surgeries s such that one of s($\mathcal{L}_0'$) and s($\overline{L_n'}$) is contained in a ball. $s(\overline{L_n'})$ can't be (it contains a leaf mapping with non-zero degree), so $s(\mathcal{L}_0')$ is contained in a 3-ball. A further sequence of surgeries can therefore reduce it to a collection of 2-spheres; so we can assume that $s(\mathcal{L}_0')$ is a collection of 2-spheres. By Reeb stability, these spheres have neighborhoods meeting only other sphere leaves, so no non-zero degree leaf can limit on any leaf of $s(\mathcal{L}_0')$. But by Lemma 2, there is a non-zero degree leaf L_i'' in every $s(L_i')$, and $\overline{L_1''}\cap\cdots\cap\overline{L_{n-1}''}\subseteq s(\overline{L_1'})\cap\cdots\cap s(\overline{L_{n-1}'})$ $= s(\overline{L_1'}\cap\cdots\cap\overline{L_{n-1}'}) = s(\mathcal{L}_0')$, so $\overline{L_1''}\cap\cdots\cap\overline{L_{n-1}''} = \emptyset$, since none of these leaves can actually limit on any leaf of s($\mathcal{L}_0'$). This is not actually a contradiction; we've reduced the number of leaves it takes to end up with the empty set, but we've done it for a different pullback lamination, $s(\mathcal{L}')$. However, we can then continue this same process, lumping together the first n-2 leaves. Eventually we will find a pullback lamination having two non-zero degree leaves with disjoint closures. But this would contradict Proposition 15. ∎

Proposition 17. *No sublamination of $\mathcal{Z}$ is essential.*

Proof: Suppose $\mathcal{L}_0'\subseteq\mathcal{Z}$ is an essential sublamination. Either $\mathcal{L}_0'\cap\overline{\mathcal{N}}=\emptyset$ or not. If $\mathcal{L}_0'\cap\overline{\mathcal{N}}=\emptyset$, surround $\mathcal{L}_0'$ by surfaces missing $\overline{\mathcal{N}}$ and surger them to spheres, bounding ball. s($\mathcal{L}_0'$) must end up contained in the balls, because s($\overline{\mathcal{N}}$) contains leaves mapping with non-zero degree. But if $\mathcal{L}_0'$ is essential, then s($\mathcal{L}_0'$) is isotopic (modulo sphere leaves) to $\mathcal{L}_0'$, a contradiction; an essential lamination cannot be contained in a 3-ball.

But if $\mathcal{L}_0'\cap\overline{\mathcal{N}}\neq \emptyset$, then $\mathcal{L}_0'\cap\overline{L'}=\mathcal{L}_1'\neq \emptyset$ for some leaf $L'\subseteq\mathcal{N}$. But then (since $\mathcal{L}_1'$ is a proper sublamination of $\overline{L'}$) f($\mathcal{L}_1'$)$\neq\mathcal{L}$, By Corollary 5. Since $\mathcal{L}_1'$ is essential, every leaf π_1-injects into M, so every leaf π_1-injects to its corresponding leaf under f. Since every leaf maps with degree zero, [B3] says that every leaf of $\mathcal{L}_1'$ is a plane or annulus. If every leaf of $\mathcal{L}_1'$ is a plane, [Ga1] implies that M is a 3-torus, hence Haken, a contradiction. So some leaf L_1' of $\mathcal{L}_1'$ is an annulus, so [B3] L_1' must map under f onto a representative of an end ϵ of the corresponding leaf L_1 of $\mathcal{L}$. So

$f(\mathcal{L}_1') \supseteq f(\overline{L_1'}) \supseteq \lim_\epsilon(L_1)$, where $\lim_\epsilon(L_1)$ is the limit set of the end ϵ of L_1, which is a non-empty sublamination of $\mathcal{L}$, hence equals $\mathcal{L}$. So f($\mathcal{L}_1'$)=$\mathcal{L}$, a contradiction.
∎

§5
Concluding remarks

It still remains to show that every leaf of $\mathcal{L}'$ maps with degree either 0 or 1. Thus far this has eluded us, but it seems that it should follow from the analysis we have presented here **and** the additional fact that our map f actually has **degree 1**. This means that if there are two inverse image leaves of a leaf L, each mapping with non-zero degree, then there are two such leaves with degrees of opposite sign. This in turn means that, 'morally', one maps orientation-preservingly, and the other orientation-reversingly. Yet they have, by Proposition 15, intersecting closures. This seems impossible.

Conjecture. *If $f{:}M\to N$ is a map of positive degree between non-Haken 3-manifolds, and $\mathcal{L}$ is an essential lamination in N, with pullback $\mathcal{L}'$, then no leaf of $\mathcal{L}'$ maps to its corresponding leaf with negative degree.*

Of course, this has 'seemed' impossible (to me) for quite some time, without being provably impossible!

§6
The (hypothetical) end

This last part is an attempt to give a very rough idea of where all these ideas might be leading. As such, we cannot make it particularly rigorous.

This conjecture would of course be wasted without some idea of how to use it to deform homotopy equivalences to homeomorphisms. There are (at least) two approaches:

(1) With the better understanding of the structure of pullbacks achieved, try to show that you can surger the pullback to make it essential, or
(2) cheat.

The 'cheat' we have in mind is to try to mimic the argument of [Br1]. That is, we pick an essential loop γ in a leaf L of $\mathcal{L}$, and look at $f^{-1}(\gamma) \subseteq f^{-1}(\mathcal{L}) = \mathcal{L}'$. The idea is that $f^{-1}(\gamma)$ is a finite collection of loops $\gamma_1,\ldots,\gamma_n$. If L' is the (conjectured) degree-one leaf mapping to L, containing the loops $\gamma_1,\ldots,\gamma_k$, say, then the sum of their degrees (thought of as mapping to γ) is one, and for all of the other leaves mapping to L, the sum of degrees of loops in each leaf is zero. This means that we ought to be able to splice together the inverse images in each leaf, and then make the resulting degree-zero loops go away; they would, after all, map null-homotopically into the loop γ. This would make the inverse image of the loop γ a single loop γ', so f induces a map f:M$\setminus\gamma'\to$N$\setminus\gamma$. This is precisely the setting of [Br1].

So we should try to show the same thing that was shown in [Br1]; namely that this map is injective on the level of π_1. This would then allow us to deform the homotopy equivalence f to a homeomorphism, as in [Br1].

It won't be, in general, of course. But we can make some more conjectures:

Conjecture. *If L' is the (hypothetical) unique leaf of $f^{-1}(L)$ mapping to L with degree one, then $f^{-1}(\gamma)\cap L'$ consists of a loop mapping with degree one, together with loops mapping with degree zero. If L'_0 is a leaf mapping to L with degree zero, then every loop of $f^{-1}(\gamma)\cap L'_0$ maps to γ with degree zero.*

The first part of this is false; 'folding' across γ can create parallel loops mapping with degree 1 and -1. But if we change 'a loop mapping with degree one' to ' parallel loops mapping with degree ± 1', then this conjecture is perhaps not altogether far-fetched.

Then we could 'clean up' everything, by splicing together loops, leaving a single degree-one loop γ' in L' without altering the complement of this remaining loop (I think - this 'cleaning up' nonsense is kind of complicated). Then the question really becomes whether or not γ' can cross infinitely many (really, really (whatever that means)) distinct surgery disks. This problem (assuming we could get this far) at least seems easier than (1).

Much of the argument of [Br1], to establish the π_1-injectivity of this restricted map, actually goes through, except where the assertions that $\pi_1(L')\rightarrow\pi_1(\mathrm{L})$ and $\pi_1(L'\backslash\gamma')\rightarrow\pi_1(\mathrm{L}\backslash\gamma)$ are π_1-injective are invoked. The first could probably be side-stepped by surgery, but the second one sounds a bit tougher. But it also doesn't seem like the second assertion should be quite as essential (pardon the pun) to the whole argument, since it is not really saying much (assuming the first can be sidestepped) about the complement of γ' in M (if you think of γ' as already being as 'simple' as possible).

This whole idea might make some people nervous - there are homotopy equivalences between 3-spheres, for example, which have a knot as the inverse image of the unknot, so the restricted map certainly cannot be π_1-injective. Nor does it seem easy to figure out how to deform the map, in those situations, so that it would be π_1-injective. But since we are looking at loops in a pullback lamination, we are dealing with (two levels of) pullbacks of codimension-one objects, which tend to be far more tractible. There is also a growing body of work showing that one can establish the sort of result we are after here, by finding the 'right' loop in a 3-manifold. This is the approach both of Casson and Jungreis' proof of the Seifert-fibered space conjecture [C-J], and Gabai's recent work on deforming homotopy equivalences for hyperbolic 3-manifolds [Ga2].

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