Knots with unique minimal genus Seifert surface and depth of knots

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Abstract. We describe a procedure for creating infinite families of knots having unique minimal genus Seifert surface which cannot be the sole compact leaf of a depth one foliation.

§0 Introduction

Thurston [Th1] has shown that every compact leaf F of a taut foliation $\mathcal{F}$ of a 3-manifold M has least genus among all surfaces representing the homology class of the surface, that is, it realizes the Thurston norm of that surface. Conversely, Gabai [Ga1] has shown that every Thurston norm minimizing surface is the sole compact leaf of a *finite depth* foliation of M. A foliation $\mathcal{F}$ has finite depth if the leaves of $\mathcal{F}$ can be partitioned into finitely many classes $\mathcal{D}_0$, $\mathcal{D}_1$, $\dots$,$\mathcal{D}_n$, where $\mathcal{D}_0$ consists of the compact leaves of $\mathcal{F}$, and each leaf in $\mathcal{D}_i$ limits only on leaves in $\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{i-1}$. The smallest n for which this is true is called the depth of $\mathcal{F}$.

These results have been successfully employed, largely by Gabai, to compute the genera of many classes of knots, such as arborescent knots [Ga2] and the knots in the standard knot tables [Ga3], by constructing finite depth foliations of the knot exteriors, with a candidate Seifert surface as sole compact leaf. For each of these constructions, the foliations built have depth one. The smallest depth of a finite depth foliation for a knot (with a Seifert surface as sole compact leaf) is called the *depth* of the knot; all of these knots therefore have depth (at most) one.

Cantwell and Conlon [CC1] gave the first examples of knots with arbitrarily high depth, by employing an iterated Whitehead doubling construction. The large number of non-parallel incompressible tori in such a knot complement forces the depth of the knot to be correspondingly high. They then asked whether or not every hyperbolic knot, by contrast, must have depth one.

Kobayashi [Ko] showed that the answer to this question was 'No', by exhibiting a hyperbolic knot, having a unique mininal genus Seifert surface, which cannot be the leaf of a depth one foliation. It is, however, the leaf of a depth two foliation, and so Kobayashi's knot has depth two.

In this paper we show that Kobayashi's example is not alone; we construct large numbers of knots with unique minimal genus Seifert surface which cannot be the compact leaf of a depth one foliation of the knot exterior. Our approach is to

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focus on knots with free Seifert surfaces of genus one, that is, Seifert surfaces whose complements are handlebodies of genus two. By focusing on surfaces of genus one, we can employ a necessary condition for the surface to be a leaf of a depth one foliation, due to Cantwell and Conlon [CC2]; by using free Seifert surfaces, we can check algebraically that their condition is not satisfied, using Beiri-Neumann Strebel invariants [BNS] and a computational tool due to Brown [Bro]. Finally, we can use some standard cut and paste techniques, and the algebra of normal forms in a free group, to show that many of the examples we construct have unique minimal genus Seifert surface.

§1
The construction

Our examples are based on the construction of knots with genus one free Seifert surfaces, given in [Br1]. The basic idea is to start with a 'base' knot K with genus one free Seifert surface F, and then choose a simple loop γ lying on the boundary of $(S^3\backslash\text{int}N(F)) = X(F)$, which bounds a disk in $N(F)$; see Figure 1. We choose curves γ lying on the four-punctured sphere shown there; such curves will always bound disks in $N(F)$. Because γ is unknotted in $N(F)$, hence in S^3, $1/n$ Dehn surgery on γ will yield S^3 back again. The knot K, and the Seifert surface F, are carried under the surgery to a new knot and Seifert surface K_n and F_n in $\gamma(1/n) = S^3$. By [Br1, pp. 63-64], the surfaces F_n are all free.

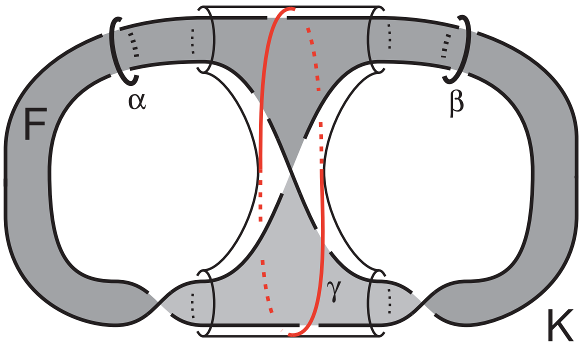


Figure 1

In anticipation of the next sections, to get the additional properties on K_n and F_n we desire, we will also impose three additional conditions on γ (and K).

First, $1/p$ and $1/q$ surgeries along the curves α and β labelled in Figure 1 will simply add full twists to K, disjoint from the region of $\partial X(F)$ where we will be choosing γ, and so will not much affect the construction. We can, in fact, think of ourselves as carrying out a <u>family</u> of constructions, one for each initial knot $K(1/p, 1/q) = K_{p,q}$. For notational convenience, we will routinely suppress the fact that K_n really also depends on p and q, that is, that the base knot K we are working with really depends on these two parameters. We need to keep these extra parameters in mind, however, because large numbers of twists (where here large essentially means $p, q \geq 2$) will, we shall see, insure that the resulting knots K_n have unique minimal genus Seifert surfaces. We assume the twists are added so that they add to, rather than cancel out, the half-twists already present in K. Our first condition, therefore, is that our construction of the knots K_n is actually built upon the base knots $K_{p,q}$.

Ultimately we will verify nearly all of the properties we need to establish about K_n and F_n by making computations in $\pi_1(X(F_n)) = F(a,b)$, the free group on two letters a and b. This task will be much easier using a picture of $X(F)$ as a standard handlebody, with K, α, β, and γ drawn on its boundary (where "K" here really means a curve, isotopic to K, on $\partial X(F)$ which cobounds an annulus with K in $N(F)$). By choosing the "obvious" (vertical) compressing disks as the cores of our 1-handles in $X(F)$ to serve as the standard (horizontal) cores of the 1-handles for our handlebody, we obtain Figure 2 (see [Br1]). We have drawn the curve γ chosen in Figure 1, but a very large class of curves lying in the 4-punctured sphere complementary to α and β will work for our purposes. In some sense, the best approach in fact is to find the necessary curve γ in this standard picture, and then determine which curve on $\partial X(F) \subseteq S^3$ it came from.

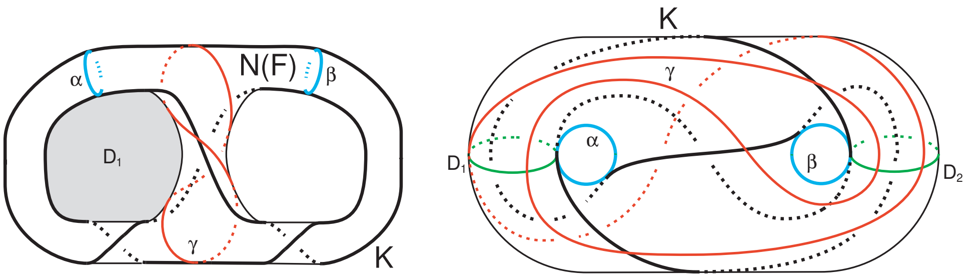


Figure 2

Throughout the rest of the paper, we will follow the standard practice of representing the inverses of the generators a, b of the free group $F(a,b)$ by A and B, respectively, and the inverse of a word w in the letters a, b by $\overline{w}$.

For our second condition, we choose a γ which is homotopically essential in $X(F)$, but is null-homologous in $\partial X(F)$; that is, γ separates $\partial X(F)$ into two once-punctured tori. In terms of our standard picture of $N(F)$ and γ, this is straightforward to verify; we simply need to check that in the four-punctured sphere in $\partial X(F)$ obtained by omitting the two handles, in our figure, γ does not separate the ends of either handle. This is sufficient, because γ does separate the four-punctured sphere. This condition can also be readily established from the word in $F(a,b)$ representing γ; the exponent sums of a and b in the word must both be zero.

When we straighten out the handlebody $X(F)$ to a standard handlebody, the knot K and the loop γ are carried to simple loops in $\partial X(F)$. We imagine pushing γ into $X(F)$ to a curve γ' on a surface parallel to $\partial X(F)$, to make it disjoint from K again. The isotopies of K (above) and γ can be thought of as a single isotopy of $K \cap \alpha \cup \beta \cup \gamma$. Because γ' bounds a disk D in S^3, the effect of $1/n$ Dehn surgery on an object disjoint from γ' is to cut the object open along the disk, give one side of the disk n full twists, and then reglue. As seen in Section 2 of [Br2], since the disk D may be chosen to meet $X(F)$ in an annulus connecting γ' to γ, $X(F_n)$ and $K_n \subseteq \partial X(F_n)$ can be obtained from $X(F)$ and $K \subseteq \partial X(F)$ by simply setting $X(F_n) = X(F)$ and applying n Dehn twists to K along the curve γ, to obtain K_n (Figure 3).

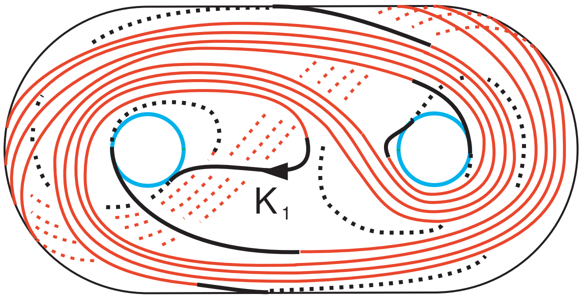


Figure 3

Because we start with a non-trivial knot K - in our examples, using $p, q \geq 2$, the knots K we start with are the 2-bridge knots [Schu] with continued fraction expansion $[2p+1, -(2q+1)]$ - the surface F, being genus one, must be incompressible, and so $\partial X(F) \setminus K$ is incompressible in $X(F)$. By applying a criterion of Starr [Sta], we can also see this directly, since there is a system of cutting disks for $X(F)$, whose boundaries split $\partial X(F)$ into a pair of thrice-punctured spheres P_1, P_2, so that K meets each P_i in essential arcs which join every pair of boundary curves. In our case, we can use the standard set of disks D_1, D_2, D_3 where D_1 and D_2 are the disks chosen above, and D_3 is the horizontal disk in the middle of the handlebody in Figure 2b.

Our third condition on γ is that it meet K and the cutting disks for $X(F)$ so that Dehn twisting K along γ' yields a loop K_1 meeting the pair of thrice-punctured spheres in essential arcs. In particular, this means that γ itself cannot have a trivial arc of intersection with either of the three-punctured spheres P_i (Figure 4). Note that this condition is meant to apply to γ *as drawn*, and not up to isotopy. This condition on γ is not invariant under isotopy.

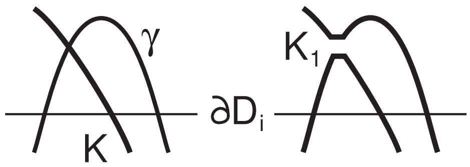


Figure 4

Since the arcs of intersection of K_1 with the P_i basically include the arcs from K, K_1 meets each three-punctured sphere on all possible essential arc types, and so we immediately have that $\partial X(F_1) \setminus K_1$ is incompressible in $X(F_1)$, and so K_1 is non-trivial in S^3 and F_1 is a least genus Seifert surface for K_1.

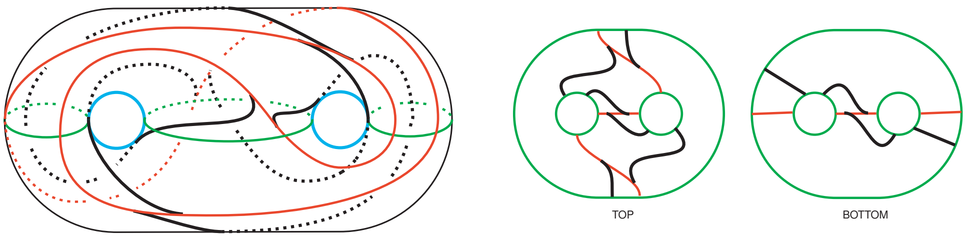


Figure 5

This condition is best determined by building the train track in $\partial X(F)$ which carries both K and K_1; this amounts to turning each point of intersection of K with γ' into a pair of switches (Figure 5). We can then see directly that the intersection of the train track with each three-punctured sphere does not carry a trivial arc, so long as it does not carry an *outermost* one. This is essentially an Euler characteristic argument. No horizontal boundary component of the train track is boundary parallel, and no complementary region is a smooth disk or monogon. Therefore, no complementary region has strictly positive (orbifold) Euler characteristic, and so no union of regions can.

We provide several examples of such γ' in Figure 6. Note that a Dehn twist in the opposite direction yields a knot which does have trivial arcs, while twisting n times in the direction in which one twist works yields knots K_n which also meet the punctured spheres in essential arcs, because the resulting curves are carried by the same train track as K_1. K_n is therefore non-trivial in S^3, and F_n is least genus for K_n, for each $n \geq 1$.

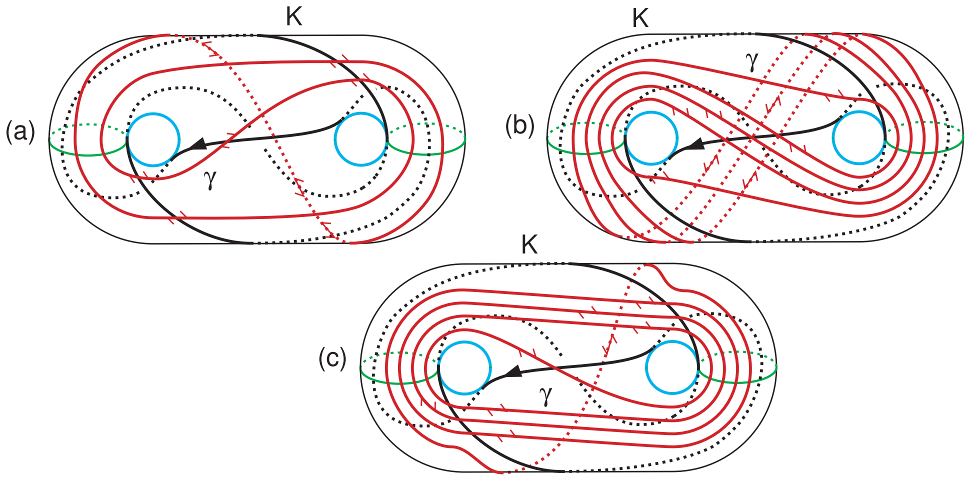


Figure 6

The fundamental group of $X(F)$ is free of rank two; it has a basis represented by loops, which we may take to lie in $\partial X(F)$, and which each intersect exactly once one of the two standard compressing disks D_1, D_2 for $\partial X(F)$ labelled in Figure 2. The loop which intersects the left disk, oriented to be travelling down as it passes through the disk, we will denote a; the other, oriented similarly, will be denoted by b. An element x of $\pi_1(X(F))$ can be written as a word in a and b and their inverses; this word can be read off from a loop representing x by reading off the sequence of disks D_1 and D_2, and the directions, that x passes through.

In the end, it will not matter much which specific words in $F(a,b)$ our knots K_n spell out; our arguments turn out to be fairly general, relying only on the properties outlined in Section 1. But with a little practice, it becomes a straightforward exercise to determine the word representing the curve K_n, after $1/n$ surgery on γ, $1/p$ surgery on α, and $1/q$ surgery on β. Note that an ordering for these surgeries does not need to be given; because these curves bound disjoint disks in S^3, surgery on each is essentially independent of the others. The basic idea is to first read off the word representing γ, as we traverse the curve. Then we read off the word representing $K_{p,q}$ (which is, in fact, $a^{p+1}b^{q+1}A^{p+1}B^{q+1}$), but each time we cross γ, we insert n copies of a cyclic conjugate of γ or $\overline{\gamma}$ into the word being read. Determining which cyclic conjugate is a matter of bookkeeping, keeping track of where, as γ is being read off, $K_{p,q}$ is crossing γ.

For example, the words read off by the curves in Figures 3, 6(a), 6(b), and 6(c) are, in order

$A^{p+1}(baBA)^n b^{q+1}(bABa)^n a^{p+1}(BAba)^n B^{q+1}(BabA)^n$

$A^{p+1}(BabA)^n(baBA)^n(bABa)^n b^{q+1}(aBAb)^n a^{p+1}(bABa)^n(BAba)^n(BabA)^n B^{q+1}$ $(AbaB)^n$

$A^{p+1}(babaBABA)^n b^{q+1}(bABABaba)^n(baBABAba)^n(babABABa)^n a^{p+1}$ $(BABAbaba)^n B^{q+1}(BababABA)^n(BAbabaBA)^n(BABababA)^n$

$A^{p+1}(bAbaBaBA)^n(bABaBabA)^n(baBaBAbA)^n b^{q+1}(bAbABaBa)^n a^{p+1}$ $(BaBAbAba)^n(BabAbABa)^n(BAbAbaBa)^n B^{q+1}(BaBabAbA)^n$

where we have read each word starting at the black triangle, reading in the direction that the triangle indicates.

Note that because the curves $\gamma \subseteq \partial X(F)$ that we choose are disjoint from α and β, the word we read off in $F(a,b)$) for γ never has a repeated letter. This is because for a repeated letter to occur (b, say), γ must pass through the disks D_2, D_3, and D_2, in order. This has the effect of "trapping" γ against β (Figure 7), which then requires γ to pass through D_2 in the opposite direction before passing through D_1, giving an inessential intersection of γ with P_2, contradicting our third assumption. The other possible repeated letters are similar. Note also that the cyclic conjugates are never inserted within one of the powers, for the same reason; the high powers arise within small neighborhoods of α and β, which γ does not meet.

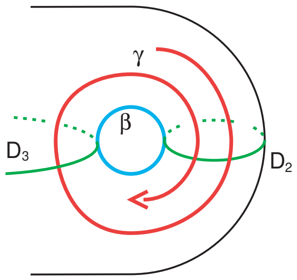


Figure 7

§2
Depth greater than one

We now show that for curves γ satisfying the conditions above, the knot K_n obtained by $1/n$ surgery along γ, *for* $n \geq 2$, has free genus one Seifert surface F_n which is not the sole compact leaf of a depth one foliation. The key ingredient to showing this is the following result, due to Cantwell and Conlon. Here M denotes the manifold obtained by gluing a 2-handle to $X(S)$ along the annulus $Q = X(S) \cap \partial(N(K)) \subseteq \partial X(S)$

Theorem [CC2]. *: If K is a genus-1 one knot and S is a minimal genus Seifert surface for K, then every depth one foliation of $X(K)$, with S as sole compact leaf, induces a fibration on M, transverse to ∂M and to a core of the attached 2-handle.*

Conversely, such a fibration induces a depth-one foliation on $X(K)$ with S as sole compact leaf.

Therefore, to show that our Seifert surfaces F_n for K_n are not the sole compact leaves of a depth one foliation, it suffices to show that the manifold obtained by gluing a 2 handle to $X(F_n)$ along $X(F_n) \cap \partial(N(K_n)) \subseteq \partial X(F_n)$ is not a surface bundle over the circle. To show this, we will use the Bieri-Neumann-Strebel invariant [BNS] of the fundamental group of $X(F_n) \cup_Q$(2-handle))$= M_n$. In the setting of 3-manifold groups, the BNS invariant is essentially the same as the fibered faces of the unit ball in the Thurston norm [Th1]; that is, the BNS invariant is an open subset in the unit sphere of $H_1(M_n;\mathbb{R}) = \mathrm{Hom}(\pi_1(M_n),\mathbb{R})$ whose points represent homomorphisms of $\pi_1(M_n)$ to $\mathbb{Z}$ having finitely generated kernel. By the Stallings Fibration Theorem [Stl], such a homomorphism is represented by a bundle map from M_n to the circle (and conversely). Therefore, if the BNS invariant is empty, there are no homomorphisms from $\pi_1(M_n)$ to $\mathbb{Z}$ with finitely generated kernel, so M_n is not a bundle over the circle.

$\pi_1(M_n)$ is, by the Seifert-Van Kampen Theorem, a one-relator group, since M_n is obtained from the genus-2 handlebody (with free fundamental group) by adding a single 2-handle (giving the relator). The relator is the word, in the generators a and b of the free group, representing the knot K_n, since K_n is the core of the gluing annulus for the 2-handle.

Brown [Bro] has produced an algorithm for computing the BNS invariant of a 1-relator group $G = < a,b \mid R >$. His method requires that the relator R have trivial abelianization; this will be the case for our groups, since K_n bounds a punctured torus in $\partial X(F)$, so is null-homologous. K_n is therefore trivial in the abelianization of $\pi_1(X(F))$, since this group is canonically isomorphic to the first homology of $X(F)$. It also requires that the relator R be reduced and cyclically reduced (i.e, does not have an adjacent pair, and does not start and end with a pair, of letters a, A or b, B). This is also the case in our setting, because $K_n|(D_1 \cup D_2 \cup D_3)$ consists of essential arcs in $\partial X(F_n)|(D_1 \cup D_2 \cup D_3)$. In our setting, the BNS invariant of G will be a subset of the unit sphere in $H_1(M_n;\mathbb{R})$. $H_1(M_n;\mathbb{R})$ can be identified with $\mathrm{Hom}(\pi_1(M_n),\mathbb{R}) = \mathbb{R}^2$, since $\mathrm{Hom}(\pi_1(M_n),\mathbb{R}) = \mathrm{Hom}(F(a,b),\mathbb{R})$, because the relator K_n, being null-homologous, is automatically sent to 0.

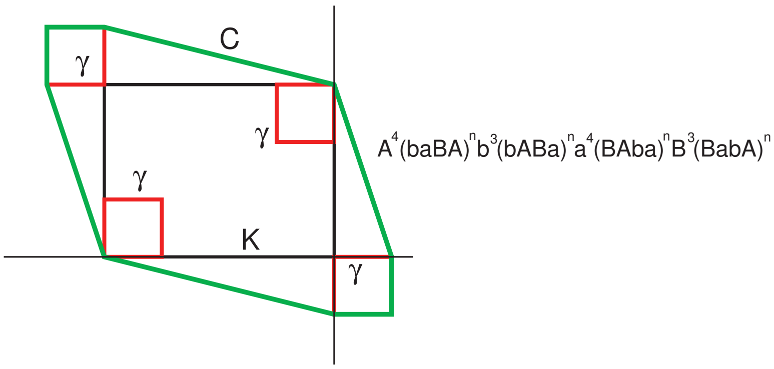


Figure 8

Brown's algorithm consists of tracing out the relator R as a path in $\mathbb{Z}^2 \subseteq \mathbb{R}^2$, thinking of R as lying in $\mathbb{Z}^2$ instead of $F(a,b)$. However, we trace out R in the order in which it is written; we do not try to simplify it in $\mathbb{Z}^2$ first; see Figure 8 for an example. Let C denote the boundary of the convex hull of the traced out path; it is a finite-sided convex polygon. A vertex v of C is called *simple* if it is crossed by the path traced out by R exactly once. C will necessarily contain horizontal and vertical edges (since the relator is cyclically reduced); such an edge is called *special* if both of its endpoints are simple. Then [Bro, Theorem 4.4] the BNS invariant of G consists of open arcs in the unit circle in $H_1(M_n;\mathbb{R})$, one for each simple vertex or special edge in C, *provided*, in the second case, that the line containing the special edge intersects C only in that edge. His algorithm also describes how to compute these arcs.

For our purposes, however, we wish to establish that the BNS invariant is *empty.* We therefore wish to find knots K_n which, when thought of as words in the free group $F(a,b)$ and traced out as a path in $\mathbb{Z}^2 \subseteq \mathbb{R}^2$, have no simple vertices or special edges. That is, every vertex of C is crossed at least twice by K_n. For curves γ and K chosen as above, this is completely routine, provided we take $n \geq 2$, that is, we spin K at least twice around γ. The basic idea is that, since γ is null-homologous in $\partial X(F)$, and therefore in $X(F)$, when K_n is traced out, each occurrence of the cyclic conjugates of γ and $\overline{\gamma}$ traces out a loop. Tracing each out $n \geq 2$ times therefore insures that each vertex that these pieces of K_n meet have already been crossed at least twice by K_n. In fact, since with our choice of K above K_n is represented by a word of the form $A^{p+1}(\text{loops})b^{q+1}(\text{loops})a^{p+1}(\text{loops})B^{q+1}(\text{loops})$, the path traced out is essentially a large (for p,q large) rectangle, with extra loops traced at the four corners. Since every extra loop is traced out at least twice, the only possible simple vertices or special edges that can be created are in the sides of the large rectangle, where any simple vertex will in fact be contain in a special edge. But the line containing such an edge will meet C at the corners of the large rectangle, where the path has crossed at least twice. So neither criterion of Brown's algorithm can be met, and so all of the groups $< a,b \mid K_n >$ constructed in this way have trivial BNS invariant.

Therefore, for every knot K_n and free genus one Seifert surface F_n constructed in this way, the manifold obtained by gluing a 2 handle to $X(F_n)$ along $X(F_n) \cap \partial(N(K_n)) \subseteq \partial X(F_n)$ is not a surface bundle over the circle, provided that $n \geq 2$. Therefore we have:

Proposition 1. *For curves γ satisfying the conditions of Section 1, and for every $n \geq 2$, the surface F_n cannot be the sole compact leaf of a depth one foliation of $X(K_n)$.*

§3
Unique genus one Seifert surface

We now give a procedure for verifying that a Seifert surface like the ones constructed above are the unique minimal genus spanning surface for the knot. As in the previous section, the basis for the procedure is geometric, but we will ultimately verify it algebraically, using the the word in the free group $F(a,b)$ that the knot K_n spells out.

Our starting point is a result of Scharlemann and Thompson, which says, essentially, that to rule out a second non-isotopic minimal genus spanning surface, we need only rule out the existence of a second such surface *disjoint* from the first:

Theorem [ST]. *: If S and T are minimal genus Seifert surfaces for the knot K, then there is a sequence of minimal genus Seifert surfaces $S = S_0, S_1, \ldots, S_n = T$ such that, for each $i, 1 \leq i \leq n, |S_i \cap S_{i-1}| = K$.*

So if one of the knots K_n constructed as above had a second genus-1 Seifert surface Σ, then it would have one which was disjoint from our surface F_n. Σ is therefore contained in the handlebody $H = X(K_n)|F_n = X(F_n$, with boundary $\partial\Sigma = K_n \subseteq \partial H$. Because Σ is not isotopic to F_n in S^3, Σ is not boundary parallel in H. After isotopy, $\Sigma \cap D_3 \subseteq D_3$ consists of circles and arcs, and all circles of intersection can be removed by a standard innermost circle argument, using the incompressibility of Σ. The intersection must still be non-empty, because $\partial\Sigma = K_n$ meets ∂D_3. Choose an outermost arc η of $\Sigma \cap D_3 \subseteq D_3$, which together with an arc ω of ∂D_3 bounds a disk $\Delta \subseteq D_3$ whose interior is disjoint from Σ. η cannot be ∂-parallel in Σ. For otherwise when we look at the intersection of the disk E cut off by η (with boundary $\eta \cup \eta_0$) with the cutting disks D_1 and D_2, an outermost (i.e, furthest from η) arc of intersection will cut an arc off of η_0, lying in K_n, whose endpoints lie in D_i, $i = 1$ or 2. But this contradicts our third condition on K_n. η therefore is non-separating on Σ, and so when we ∂-compress Σ along Δ, we obtain an annulus A_1 (Figure 9).

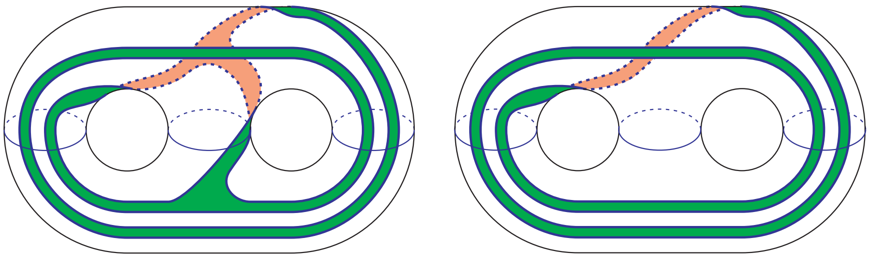


Figure 9

ω lies in one of the two punctured tori $T_1 \cup T_2 = (\partial H))|K_n$, say T_1. ω doe not separate T_1, because it is essential in T_1; the argument is identical to the one given above for Σ. $T_1|\omega$ is therefore an annulus A_2 with (we may assume) $\partial A_2 = \partial A_1$. $A_1 \cup A_2 = T \subseteq H$ is then an torus (it cannot be a Klein bottle, because handlebodies do not contain non-orientable closed surfaces), which must be compressible. Because H is irreducible, T either is contained in a 3-ball, so the curves $c_1 \cup c_2 = A_1 \cap A_2$ are null-homotopic in H, or T bounds a solid torus M_0 in H. The first case is in fact impossible, because the curves c_i are (non-separating, hence) essential in T_1, which is incompressible. In the second case, if c_1 (say) generates $\pi_1(M_0)$, then A_1 is parallel to A_2 through M_0, that is, A_2 is ∂-parallel (Figure 10a). But this in turn implies that Σ is ∂-parallel, since reversing the ∂-compression along Δ, starting with a ∂-parallel annulus, yields a ∂-parallel surface; Δ must lie outside of M_0, since otherwise Σ is compressible (Figure 10b). c_1 must therefore represent a proper power of the generator of $\pi_1(M_0)$, and so represents a proper power in $\pi_1(H)$.

Therefore, if Σ exists, then for every outermost arc of $\Sigma \cap D_3 \subseteq D_3$, ∂-compression of Σ along the outermost disk of D_3 that the arc cuts off produces an annulus whose boundary components represent a proper power in the free group $F(a, b) = \pi_1(H)$.

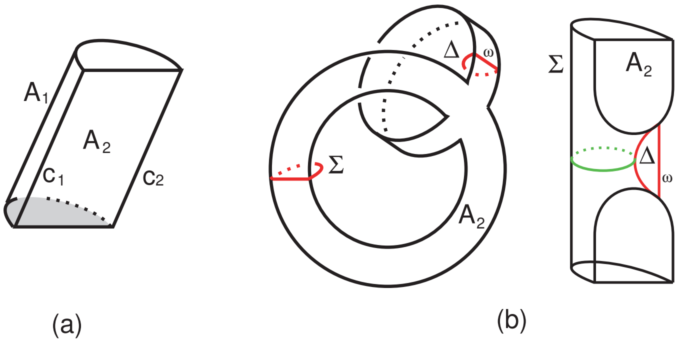


Figure 10

The main point, however, is that under this process of ∂-compression, curves representing only six words in $F(a,b)$ can be formed out of any given curve K_n, and each can be quickly checked to determine if it is a proper power. The basic idea is that, schematically, the punctured torus complements T_1 and T_2 of K_n in ∂H are built by gluing together pieces of $\partial H|(D_1 \cup D_2 \cup D_3 \cup K_n)$, and K_n cuts $\partial H|(D_1 \cup D_2 \cup D_3)$ into four hexagons and a collection of rectangles, from the third condition on γ. The rectangles occur between arcs of K_n parallel in $\partial H|(D_1 \cup D_2 \cup D_3) = P_1 \cup P_2$; two hexagons come from each of P_1, P_2, lying between non-parallel arcs of K_n. The T_i therefore each look like a pair of hexagons with pairs of edges identified through a string of rectangles. From this point of view T_i has a very natural spine consisting of two vertices, one for each hexagon, and three arcs, one for each string of rectangles (Figure 11). Each arc can be represented by a word $\lambda, \mu\nu$ in $F(a,b)$ by reading off its intersections with the disks D_1 and D_2 dual to our generators a, b. The word representing K_n, in terms of these three words, is then $\lambda\overline{\mu}\nu\overline{\lambda}\mu\overline{\nu}$.

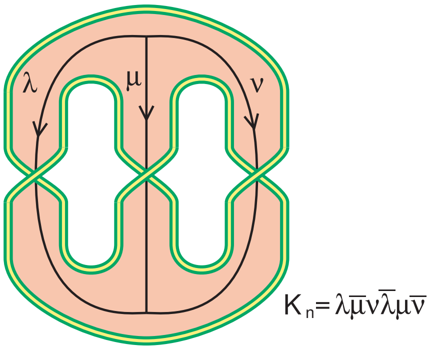
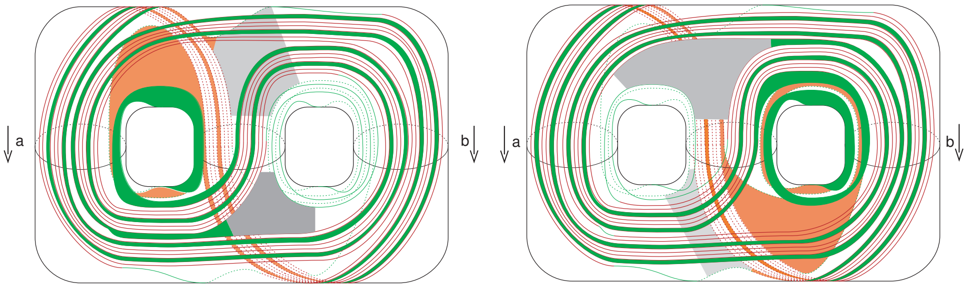


Figure 11

K_n cuts ∂D_3 into a (potentially large) collection of arcs, and each of these arcs potentially cobounds an outermost arc of $\Sigma \cap D_3$. We must therefore check that the annulus obtained by surgering T_1 or T_2, whichever one contains the arc, along each arc does not have boundaries representing proper powers. But under ∂-compression each arc will cut across one of the three strings of rectangles, resulting in an annulus made up, essentially, of the other two strings. The core of this annulus will therefore be a curve represented by one of the words $\lambda\overline{\mu}$, $\nu\overline{\lambda}$, or $\mu\overline{\nu}$, depending upon which string was cut. We should note that these words will be (cyclically) reduced; if, for example, λ begins by passing through D_1 (so starts with a) and μ passes through D_2 (so starts with b), then ν must first pass through D_3 and so first passes through D_1 or D_2 in the opposite direction, and so (is trivial or) starts with A or B. Consequently, $\overline{\nu}$ *ends* with a or b, so $\lambda\overline{\nu}$ and $\mu\overline{\nu}$ are cyclically reduced. They are, as subwords of K_n, already reduced. All other combinations of initial letters are similar.

In the end it will not be necessary to do so, but as a practical matter, finding the words λ, μ, ν is fairly straightforward. It can be done from a picture of K_n, as in Figure 12 (for one arc each of T_1 and T_2); once one of the words (for each punctured torus) is determined, the others can easily be found, since the words found will typically occur only twice (once forward and once backward) in K_n, and the remaining two pieces will have only one representation as $w_1\overline{w_2}$ and $\overline{w_1}w_2$. In practice, in fact, only two representations of K_n in the required form will typically exist at all.



(aBAb)^{n}a^p (bABa)^n (bABa)^{n}b^q(aBAb)^n

Figure 12

In the example of Figure 6(a), for example, we find that

$$K_n = [A^p][A(BabA)^n(baBA)^n][(bABa)^n b^{q+1}(aBAb)^n][a^p][a(bABa)^n(BAba)^n]$$
$$[(BabA)^n B^{q+1}(AbaB)^n]$$
$$= \lambda\overline{\mu}\nu\overline{\lambda}\mu\overline{\nu}$$

and

$$K_n = [(AbaB)^n A^{p+1}(BabA)^n][(baBA)^n(bABa)^n b][b^q][(aBAb)^n a^{p+1}(bABa)^n]$$
$$[(BAba)^n(BabA)^n B][B^q]$$
$$= \lambda\overline{\mu}\nu\overline{\lambda}\mu\overline{\nu}$$

representing the two different viewpoints of K_n as ∂T_i.

As further examples, the reader can verify that in Figure 3 we have

$K_n = [A^{p+1}][b][(aBAb)^n b^q(bABa)^n][a^{p+1}][B][(AbaB)^n B^q(BabA)^n]$

so $\lambda = A^{p+1}$, $\mu = B$, and $\nu = (aBAb)^n b^q(bABa)^n$

$K_n = [a][B^{q+1}][(BabA)^n A^p(AbaB)^n][A][b^{q+1}][(bABa)^n a^p(aBAb)^n]$

so $\lambda = a$, $\mu = b^{q+1}$, and $\nu = (BabA)^n A^p(AbaB)^n$

Our goal now is to show that none of the resulting six words (three for each of the T_i) $\lambda\overline{\mu}$, $\nu\overline{\lambda}$, or $\mu\overline{\nu}$ are proper powers in $F(a,b)$. To do this we use the fact that the normal form of the nth power of a (cyclically reduced) word w in a free group is the concatenation of n copies of w. This is because the concatenation of w's is, by our assumptions, already in normal form and cyclically reduced, and words in a free group have unique normal form [MKS]. Conversely, if v is cyclically reduced, and $v = w^n$, then the normal form for w must be cyclically reduced (otherwise w^n, in normal form, is not), and so w^n is already in normal form, so v is a concatenation of w's. So we need only show that for each of the six words w_j that the above process creates, w_j is not a concatenation of a subword of w_j.

This is where we will put the condition $p,q \geq 2$ to further use. The basic idea is that there is only one occurrence of a^{p+1} and b^{q+1} in K_n, and therefore at most one occurrence of a high power of a,b,A or B, in each of the subwords $\lambda\overline{\mu}$, $\nu\overline{\lambda}$, or $\mu\overline{\nu}$ of K_n. Here "high" means "≥ 2". For $K_{p,q}$ itself, the subwords λ,μ,ν are a, A^p, b^{q+1} respectively, for one punctured torus, and a^{p+1}, B^q, b for the other (Figure 13a).

The intersections of γ with $K_{p,q}$ can be viewed (from the point of view of each of the T_i), in pairs, as crossings of the strings of rectangles for T_i (since we can view γ as missing a pair of "very thin" hexagons in T_i). The effect then on λ,μ,ν of spinning $K_{p,q}$ around γ is to append cyclic conjugates of γ and $\overline{\gamma}$ before and after the values of λ,μ,ν for $K_{p,q}$ (Figure 13b). The conjugates are not inserted within these high powers; this is because γ is disjoint from α and β, and so the high powers in the strings of rectangles for $\partial X(F_{p,q} \setminus K_{p,q}$ can be thought of as originating in a small neighborhood of α,β, when K is spun about them to produce $K_{p,q}$. Those regions of the strings of rectangles are consequently also disjoint from γ.

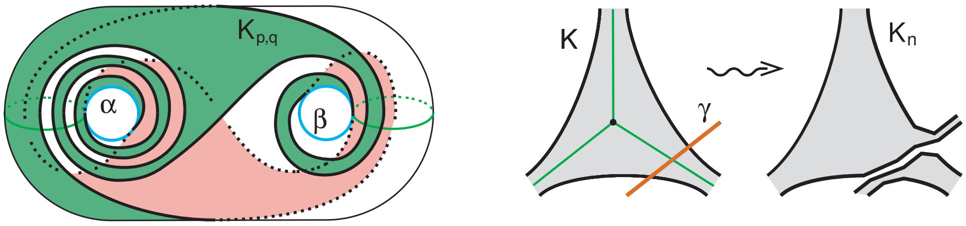


Figure 13

Because each of the subwords $w = \lambda\overline{\mu}$, $\nu\overline{\lambda}$ and $\mu\overline{\nu}$ contain, by inspection above, at least one (and therefore exactly one) high power of a,b,A or B, the only way that the subword w can be a proper power is if it contains *no other letters*, because otherwise such a power would be a concatenation of a subword of w, and so would require more than one occurrence of this high power. This immediately rules out $\nu\overline{\lambda}$ and $\mu\overline{\nu}$ for the first punctured torus, and $\lambda\overline{\mu}$ and $\mu\overline{\nu}$ for the second, as candidates for proper powers. The remaining two cases, $\lambda\overline{\mu}$ for the first and $\nu\overline{\lambda}$ for the second, can be proper powers only if γ does not cross the strings of rectangles making up these two annuli. But this is impossible; the cores of these annuli each intersect one of α,β once (Figure 13a), and so if γ misses one of the cores C (which intersects α, say), then γ lies in, and is null-homologous in, $\partial X(F) \setminus (\alpha \cup C)$, which is a punctured torus isotopic to the right half of $\partial X(F)$ in Figure 13a. γ is therefore isotopic to the boundary of this punctured torus, and so is null-homotopic in $X(F)$, contradicting our second condition on γ. Consequently, we have

Proposition 2. *For every curve γ satisfying the conditions of Section 1, and for every $p,q \geq 2$, the knots K_n constructed above have unique minimal genus Seifert surface F_n; this surface is genus 1 and free.*

§4
Hyperbolicity

We now show that for each choice of γ given in section 1, infinitely many of the resulting knots K_n are hyperbolic. To do this we demonstrate that for every

$p, q \geq 2$ the two component link $L = K_{p,q} \cup \gamma$ has hyperbolic complement. Then by Thurston's Hyperbolic Dehn Surgery Theorem [Th2], all but finitely many $1/n$ Dehn fillings along γ yield hyperbolic knot complements. Since the volumes of the complements of these knots K_n, for high values of n, are less than, but converge to, the volume of the complement of $K_{p,q}$ (whose volumes, in turn, converge to that of the three component link $K \cup \alpha \cup \beta$), infinitely many of the knot complements are distinct. Therefore [GL] infinitely many of the knots are distinct.

Hyperbolicity can be verified by showing that the topological hypotheses of Thurston's Geometrization Theorem [Th3] hold, that is, that L is not a split, satellite, or torus link. The basic outline of the proof of this follows section 3 of [Br1]. We let $F_{p,q}$ denote the genus 1 Seifert surface of $K_{p,q}$ obtained by spinning F along α and β.

Proposition 3. *The link L is not split.*

Proof: The loop $\gamma \subseteq X(F_{p,q} \subseteq X(K_{p,q})$ represents a non-trivial word in $\pi_1(X(F_{p,q})) = F(a,b)$, by the second condition imposed on γ above. Since $F_{p,q}$ is incompressible in $X(K_{p,q})$, $\pi_1(X(F_{p,q}))$ injects into $\pi_1(X(K_{p,q}))$, and so γ is non-trivial in $\pi_1(X(K_{p,q}))$. γ is therefore not contained in a 3-ball in $X(K_{p,q})$. But then for any 2-sphere S^2 in $X(L) \subseteq S^3$, whichever 3-ball complementary region (in S^3) of S^2 contains $K_{p,q}$ also contains γ. The other 3-ball region is therefore contained in $X(L)$. So L is not split. ∎

Proposition 4. *The link L is not a torus link.*

Proof: The knot $K_{p,q}$, since $p, q \geq 2$, is a non-torus alternating knot, and so by Menasco [Me] is hyperbolic. L therefore contains a non-torus knot, and so is non-torus. ∎

Lemma 5. *There is no essential annulus Q in $X(F_{p,q}) \setminus int\ N(\gamma) = M$ with one boundary component on $N(\gamma)$ and the other on $\partial X(F_{p,q}) \setminus \partial N(K) = F_0 \cup F_1$.*

Proof: For any such annulus Q, $Q \cap \partial N(\gamma) = \omega$ is a loop representing some slope u/v on the boundary torus (using the standard meridian/longitude coordinates). The other boundary component is on F_i, $i = 0$ or 1. $v \neq 0$ since otherwise Q capped off with a meridian disk is a disk D in M, with boundary in F_1, which is incompressible. ∂D therefore bounds a disk in F_1, and so D is ∂-parallel, by the irreducibility of M. In particular D separates M, but intersects the curve γ once, a contradiction. Therefore $v \geq 1$ and Q may then be capped of by an annulus in $N(\gamma)$ to (possibly a power of) the core of this solid torus, and so represents a homotopy between γ^v and a simple loop on F_1.

The manifolds $X(F_{p,q})$ are all handlebodies, as we have seen, and we can in fact think of them as the <u>same</u> handlebody, with a different curve $K_{p,q}$ drawn on them, obtained by spinning our orginal K, in our picture of $X(F)$, around α and β. From this point of view the curve γ represents the same word in all of the $\pi_1(X(F_{p,q})$, and via Q is conjugate to a word in $i_*(\pi_1(F_0))$ or $i_*(\pi_1(F_1))$ in $\pi_1(X(F_{p,q}))$. But from Figure 13a we may compute that these two subgroups are freely generated by the words $\{a^{p+1}, b^{q+1}A\}$ and $\{a^{p+1}B, b^{q+1}\}$, respectively. Consequently, if $p, q \geq 2$, then words in normal form in $i_*(\pi_1(F_0))$ have b's occurring in clumps; no word in normal form can have a single b or B surrounded by a combination of a's and/or A's. Similarly, no word in normal form in $i_*(\pi_1(F_1))$ can have a single a or A

surrounded by a combination of b's and/or B's. But for any curve γ satisfying the conditions of Section 1, the word representing γ, by the comment at the end of Section 1, will be cyclically reduced and will not have an occurrence of a^2, A^2, b^2, or B^2 in any cyclic conjugate. The same will therefore be true of γ^v, and so γ^v cannot be conjugate to a word in either of these two subgroups. ∎

Proposition 6. *L is not a satellite link.*

Proof: The argument parallels much of the argument of Proposition 3 of [Br1]; where the details are identical, we omit them. For ease of notation, we will write K for $K_{p,q}$ and F for $F_{p,q}$. In the discussion above, γ was portrayed as lying on $\partial X(F)$; in the present discussion, we think of it rather as lying slightly in the interior of $X(F)$.

Suppose that T is an incompressible torus in $X(L)$ that is not ∂-parallel. Then since $F \subseteq X(L)$ is incompressible, after disk swapping we may assume that $T \cap F \subseteq F$ consists of two families of parallel loops, one parallel to ∂F, and one parallel to a non-separating loop in F. Repeatedly pushing T across the annulus between the outermost loop of the first family and ∂F yields a new essential torus (using Lemma 5 above in place of Lemma 1 of [Br1]), which we still call T; after these isotopies, we may assume that the first family of loops is empty. All loops of $T \cap F$ are then parallel to a single non-separating loop on F.

Suppose that we still have $T \cap F \neq \emptyset$. For homological reasons (T separates $X(K)$) there must be an even number of loops of $T \cap F$, cutting F into an odd number of annuli $A_1, \dots A_k$ and a single punctured annulus P. Because $X(K)$ is hyperbolic, T must be compressible or ∂-parallel in $X(K)$. It cannot be ∂-parallel, because then P would lie in the product region between T and $\partial X(K)$, so P cannot inject into the product, on the level of π_1, implying that F is compressible. Since $T \subseteq X(K) \subseteq S^3$, T bounds a solid torus M_0 in S^3 on at least one side. T therefore either bounds the solid torus M_0 in $X(K)$, if $K \cap M_0 = \emptyset$, or $K \subseteq M_0$ and is disjoint from a meridian disk D of M_0 (which is the only compressing disk T has, unless T bounds a solid torus on both sides).

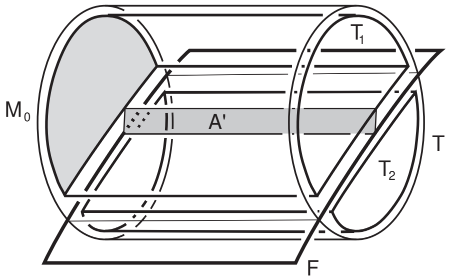


Figure 14

If $M_0 \subseteq X(K)$, then at least one of the annuli A_i lies in M_0; choose one that is outermost in M_0. Then $\partial N(T \cup A_i)$ consists of two tori T_1 and T_2, which lie in $X(L)$, since γ is disjoint from F (Figure 14). Because A_i is outermost, one of these tori, say T_1, is disjoint from F. The argument of Proposition 3 of [Br1] applies without change to show that T_1 cannot be ∂-parallel in $X(L)$, and if T_1 is compressible in $X(L)$, then it bounds a solid torus in $X(L)$, and the solid torus may be used to isotope T in $X(L)$ to reduce the number of curves of $F \cap T$. Therefore either T_1 is essential in $X(L)$ and disjoint from F, or we can reduce the number of circles of intersection of T with F.

If $K \subseteq M_0$, then the argument of Proposition 3 of [Br1] applies to show that $T \cap F$ consists of more than two components, so one of the annuli A_i lies in M_0. Then we may repeat the argument of the previous paragraph to find either an essential torus T_1 disjoint from F or reduce the number of intersections of T with F.

Therefore, by repeatedly applying these arguments, we eventually obtain an essential torus in $X(L)$ (which we will still call T) disjoint from F. We then look at the intersection of T with the annulus Q lying between γ and $\partial N(F)$ (Figure 15) which is the intersection with $X(F)$ of a disk D in S^3. After isotopy in $X(L)$, by the incompressibility of T we may assume that the intersection consists of loops essential in Q. If $T \cap Q \neq \emptyset$, choose an outermost loop of intersection, cutting off an annulus Q' from Q with one boundary component equal to γ. By isotoping T across Q' (and therefore across γ), we obtain a new torus T' in $X(L)$.

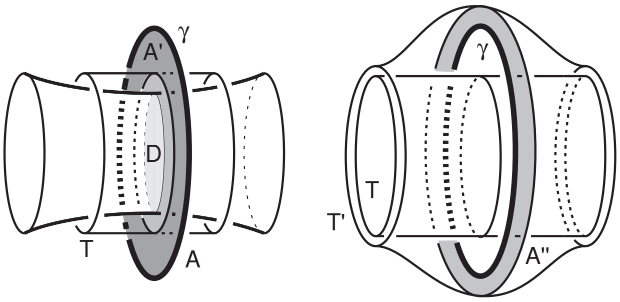


Figure 15

If T' is parallel in $X(L)$ to $\partial N(\gamma)$, then pushing T' back across γ via the annulus Q'' to T demonstrates that T bounds a solid torus in $X(L)$, a contradiction. If T' is parallel in $X(L)$ to $\partial N(K)$, then since $T' \cap F = \emptyset$, F lies in the product region between T' and $\partial N(K)$, implying that F is compressible, a contradiction. If T' is compressible in $X(L)$ via a compressing disk D, then as before either T' bounds a solid torus M_1 in $X(L)$ or it bounds a solid torus M_1 containing K or γ, and the curve is disjoint from a meridian disk for M_1.

If $M_1 \subseteq X(L)$, then $Q'' \cap M_1 = \gamma_1$ must represent a generator of $\pi_1(M_1)$. For otherwise either γ_1 is a meridian curve, and then Q'' together with a meridian disk is a disk in $X(L)$ with boundary γ, implying that γ is null homotopic in $X(K)$, a contradiction, or γ_1 represents a proper power of the core of M_1. But then since $M_1 \cup Q'' \subseteq X(F)$, this implies that γ represents a proper power in $\pi_1(X(F)) = F(a,b)$, and so, when written in normal form as a word in a and b, it is a proper power of a subword. γ, however, is parallel, via the annulus Q, to a simple curve η in $\partial X(F) = G$, which, by a previous argument, cannot be a proper power in $\pi_1(G)$. This in itself is not a contradiction, since there are simple curves in G which are proper powers in $\pi_1(X(F))$. But η is disjoint from the loops α and β of Figure 1 above, and this implies that when written in the generators a, b, c, d of $\pi_1(G) =< a, b, c, d : (AcaC)(dbDB) = 1 >$ portrayed below (Figure 16), it can be chosen to be a word in only a and b, since $G \setminus (D^2 \cup \alpha \cup \beta)$ deformation retracts onto the subgraph generated by a and b of the spine of $G \setminus D^2$. The inclusion-induced map from $\pi_1(G)$ to $F(a,b)$ is the obvious one which sends c and d to 1; therefore,

η, when written as a word in a and b in $\pi_1(G)$, is represented by the *same* word in $F(a,b)$. Since this word, in $F(a,b)$, is hypothesized to be a proper power, η therefore must represent a proper power in $\pi_1(G)$, a contradiction. But then when we push T' back to T via Q'', γ becomes the core of the solid torus M_1, implying that T is parallel to $\partial N(\gamma)$, a contradiction.

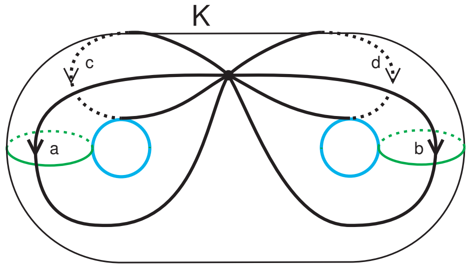


Figure 16

If T' bounds a solid torus M_1 containing K or γ, with this curve disjoint from a meridian disk D for M_1, then the argument of Proposition 3 of [Br1] applies to show that D can be made disjoint from Q''. Then when we push T' back to T via Q'', the compressing disk D persists, so T is compressible, a contradiction.

Therefore, T' must be an essential torus in $X(L)$. Continuing this process of pushing the torus across γ, we eventually find a torus T essential in $X(L)$ which is disjoint from both F and Q. But such a torus then lies in $X(F \cup Q)$, which is a handlebody, and so there is a compressing disk D' for T in $X(F \cup Q) \subseteq X(L)$. T is therefore compressible in $X(L)$, a contradiction. So there are no essential tori in $X(L)$. ∎

Consequently, the links $K_{p,q} \cup \gamma$, where $p, q \geq 2$ and γ satisfies the conditions set forth in the previous sections, is a hyperbolic link. Therefore, for infinitely many n, the knots K_n are hyperbolic. Taking this and the results of the previous two sections together, we have:

Theorem 7. *The are infinitely many hyperbolic knots K in the 3-sphere which admit genus-one, free, Seifert surfaces which cannot be the compact leaf of a depth-one foliation of the exterior of K.*

§5
Concluding remarks

In this paper we have found families of hyperbolic knots with depth greater than one. "Found" here must, however be interpreted in the weak sense; by appealing to the Hyperbolic Dehn Surgery Theorem, we can conclude that our knots K_n are hyperbolic for most values of n, but we can conclude this for no *specific* value of n. Experimental evidence, by way of snappea [We] or Snap [Go], would suggest that K_n is in fact hyperbolic for ($p, q \geq 2$ and) all $n \geq 1$; sufficient diligence might allow for a unified proof of this. We do not undertake such an effort here.

Our result falls short of the standards set by Kobayashi's result in another way, in that we do not determine the actual depth of the knots K_n that we have found. We only establish that it is at least 2, for each. Kobayashi determined that the depth of his knot, which we will call K_{ob}, is exactly 2, by constructing a sutured manifold decomposition [Ga1] of $X(K_{ob})$ by hand. We should note, however, that his knot K_{ob} is distinct from all of ours, since the unique genus-1 Seifert surface for K_{ob} does not have handlebody complement.

It is tempting to conjecture that all of the knots built here have depth 2, since they are built from (2-bridge, hence) depth 1 knots by spinning around a loop γ which bounds a surface Σ in $X(F)$. Σ is also a natural candidate for the surface to next decompose the sutured manifold $(X(F_n), \partial X(F_n) \cap N(K_n))$ along. The reader can verify, however, that this will not yield a taut sutured manifold; the surface Σ can be chosen to be one of the once-punctured torus components of $\partial X(F_n) \setminus \gamma$, and any choice of orientations yields trivial sutures under the decomposition. Unfortunately, the technology does not yet exist to show that a hyperbolic knot has depth greater than 2. To show that $X(K_n)$ does not admit a depth 2 foliation, we essentially need to show that there is no decomposing surface Σ for $X(F_n)$ such that decomposing $(X(F_n), K_n)$ along Σ, to (M', γ'), yields a sutured manifold which admits a depth 1 foliation. In our context, where $X(F_n)$ is a tuat sutured handlebody, some kind of approach to this might be possible, since every further decomposing surface Σ, being least genus and therefore incompressible, will split $X(F_n)$ into another taut sutured handlebody; M' injects, on the level of fundamental group, into the free group $\pi_1(X(F_n))$.

For our purposes, we can therefore focus on developing (more general) conditions to decide that a sutured handlebody deos not admit a depth 1 (or higher depth) foliation. For sutured handlebodies (H, A) of any genus, depth 1 implies [Ga3] that $\text{int}(H \cup_A (D^2 \times I)$ is a bundle over the circle, but when the genus of H is greater than 2, the fiber, necessarily, has infinite genus. When the genus of H is 2, the fiber has finite genus, which is why in the Cantwell-Conlon condition we may replace $\text{int}(H \cup_A (D^2 \times I)$ with $H \cup_A (D^2 \times I)$. We can, at present, offer no insight into how to overcome the problem of needing to test every possible decomposing surface.

Our techniques for establishing that K_n had unique minimal genus Seifert surface also used in a strong way the fact that the Seifert surfaces F_n had genus 1. It would be interesting to develop more general conditions along these lines for generating knots with free genus g Seifert surfaces which are unique for the knot among minimal genus Seifert surfaces. In principle is seems likely that a similar approach, using the group theoretic properties of the word representing K_n in a free group, should be successful. But the needed conditions will likely be more complicated.

Recent work of Scharlemann [Scha] implies that all of the knots K_n that we have constructed have tunnel number 2. A free genus one knot can have tunnel number at most 2; any two simple arcs in the surface F_n which split F_n to a 2-disk D form a system of tunnels for K_n, since the exterior of $K_n \cup$(tunnels) is $X(F_n)$, with the two disks in F_+ and F_- corresponding to D glued together. But Scharlemann has shown that a genus-1 tunnel-1 knot must either be a satellite knot or a 2-bridge knot, which none of our examples are; 2-bridge knots are alternating, and so have depth 1 [Ga3].

In the end, it was surprising (to the author) how many of the original collection of knots K_n had unique minimal genus Seifert surface. And so we end with the somewhat ill-posed

Question. *How common is it for a knot to have unique minimal genus Seifert surface?*

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