Small Seifert-fibered spaces and Dehn surgery on 2-bridge knots

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Abstract. We show that non-integer surgery on a non-torus 2-bridge knot can never yield a small Seifert-fibered space. In most cases, no surgery will yield a small Seifert-fibered space.

§0
Introduction

A great deal of research in 3-manifold topology in recent years has been motivated by Thurston's Geometrization Conjecture: every 3-manifold admits a canonical decomposition into pieces admitting geometries locally isometric to one of 8 'model' geometries. Further, experimental (and theoretical) evidence leads us to believe that by far the most common of the 8 geometries to occur is the hyperbolic geometry, $\mathbb{H}^3$. In particular, if M is a hyperbolic 3-manifold with boundary component a torus T, then Thurston showed [Th] that all but finitely-many Dehn fillings along T yield hyperbolic 3-manifolds.

A ready supply of examples to experiment with can be found among the classical knot exteriors $\mathbb{S}^3\backslash\mathrm{int}(\mathrm{N}(\mathrm{K}))$. In fact, by Thurston's geometrization theorem, a knot exterior $\mathbb{S}^3\backslash\mathrm{int}(\mathrm{N}(\mathrm{K}))$ admits a hyperbolic metric exactly when the knot exterior contains no incompressible, ∂-incompressible annulus (i.e., K is prime and not cabled), and contains no essential torus (i.e., K is not a satellite).

There are several ways that a manifold obtained by p/q - Dehn surgery along such a knot K can fail to be hyperbolic; the causes include the possibility that K(p/q)

(a) has finite fundamental group

(b) contains a reducing 2-sphere

(c) contains an incompressible torus

or

(d) is a 'small' Seifert-fibered space; a manifold fibered by circles, with base $\mathbb{S}^2$ and 3 multiple fibers.

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The geometrization conjecture is, in some sense, the statement that these are the only possibilities.

In light of the Thurston's Dehn-filling result, it is natural to ask under what conditions each of these other possibilities can occur. In particular, for a given (hyperbolic) knot, what restrictions can be placed on the possible Dehn surgery coefficients which lead to each of these cases? Much recent effort has been spent on these questions, and, for the first three possibilities, a great deal is now known.

Boyer and Zhang [BZ] have shown that at most 6 Dehn surgeries on a hyperbolic knot give manifolds with finite fundamental group, and the Dehn surgery coefficient p/q is either an integer or half-integer; q=1 or 2. Gordon and Luecke [GL1] have shown that at most 3 surgeries on a non-trivial knot give reducible manifolds, and p/q is integral . They have also shown [GL2] that at most 7 Dehn surgeries on a hyperbolic knot yield toroidal manifolds, and p/q is an integer or half-integer. Similar results are known, in the first and third cases, for satellite knots.

In the last case, small Seifert-fibered spaces, Miyazaki and Motegi [MM] point out that an answer to the above question can be given for satellite knots which are not cable knots: at most two, integral, surgeries can yield a small Seifert-fibered space. Nothing, however, seems to be known in the motivating case, namely hyperbolic knots. Gordon has, however, conjectured that only integral surgery on a hyperbolic knot can produce a small Seifert fibered space. In this paper, we prove that this is true, for 2-bridge knots.

Theorem. *Non-integer Dehn surgery on a non-torus 2-bridge knot cannot yield a small Seifert-fibered space.*

Our proof, unlike the others (which for the most part use the algebraic/graph-theoretical techniques found in [CGLS]), will use essential laminations. In fact, it is in some sense little more than the observation that the constructions of essential laminations in the manifolds obtained by Dehn surgery on (non-torus) 2-bridge knots, found in [De1], are incompatible with the structure theorem for essential laminations in small Seifert-fibered spaces, found in [Br1], at least if the Dehn surgery coefficient is not an integer. Consequently, one cannot obtain a small Seifert-fibered space by such a Dehn surgery.

Remark. Delman has other constructions of essential laminations in knot complements [De2], with the same properties that we exploit here, in particular for most pretzel knot complements and Montesinos knot complements; consequently, the same theorem we establish here follows immediately for these knots, as well.

There are several other ways in which to determine that a 3-manifold obtained by Dehn surgery on a knot is not a small Seifert-fibered space. Wu has shown that (because they contain incompressible surfaces, so Thurston's geometrization theorem applies) every non-trivial Dehn surgery on most algebraic knots is hyperbolic (see [Wu]). The 2π - Theorem (see [GT] or [BH]) can also prove that various ranges of Dehn surgeries on a (hyperbolic) knot yield manifolds admitting a metric of negative curvature (and so are not Seifert-fibered) - one needs to establish that the geodesic in the Euclidean structure for the cusp (i.e., the boundary torus) representing the filling has length at least 2π. However, incompressible surfaces are, in some sense, rare, so the first approach seems difficult to apply in much generality. It is also difficult to describe lengths of filling curves in terms of the standard meridian/longitude basis for the boundary torus. Essential laminations, by contrast, are (at least conjecturally, but also in practice) far more common than incompressible surfaces ('most 3-manifolds are laminar'), and in the constructions carried out so far, they in fact tend to be built in families, both of knots and of the Dehn surgeries which they 'survive'. Both of these properties lend themselves very well to the program that we are outlining here.

§1
Essential laminations in small Seifert-fibered spaces are not genuine

For basic concepts regarding essential laminations and essential branched surfaces, the reader is referred to [GO]. For the basic ideas about Seifert-fibered spaces, the reader is referred to [He]. We will assume that all 3-manifolds we consider are orientable (and hence have orientable boundary).

In [Br], we showed that every essential lamination $\mathcal{L}$ in a Seifert-fibered space M contains a sublamination $\mathcal{L}_0$ which is isotopic to a horizontal or vertical lamination; it is either transverse to all of the cicle fibers of M, or each leaf contains every circle fiber that it meets. In particular, if M is a small Seifert-fibered space, then M contains no vertical essential laminations, so $\mathcal{L}_0$ can be made horizontal. In addition, if all of the leaves of $\mathcal{L}$ are non-compact, then all leaves of $\mathcal{L}$ can be made horizontal; consequently, all components of M$|\mathcal{L}$ are products of a (non-compact) surface and I. On the other hand, if $\mathcal{L}$ contains a (necessarily horizontal) compact leaf, then the results of [Br2] show that $\mathcal{L}$ again has only I-bundle complementary components.

Gabai [Ga1] has introduced the terminology *genuine* to describe an essential lamination which has a complementary component which is not an I-bundle. The lamination is a genuine one, because it is not 'just' a foliation which has been split open along some collection of leaves. We can therefore rephrase the results described above to say that an essential lamination in a small Seifert-fibered space is never genuine.

We will apply this observation, in the next section, by exhibiting some genuine essential laminations, in manifolds obtained by Dehn surgery on 2-bridge knots. To do this, we must be able to recognize a non-I-bundle complementary component of a lamination. This can be done fairly readily, however, using an essential branched surface which carries the lamination.

Proposition. *If $\mathcal{L} \subseteq M$ is an essential lamination, carried with full support by the essential branched surface B, then $\mathcal{L}$ is* **not** *genuine if and only if every component of M$\setminus$int(N(B)) is an I-bundle over some compact surface Σ, with corresponding ∂I-bundle equal to $\partial_h N(B)$, and $\partial_v N(B)$ equal to the I-bundle over $\partial\Sigma$.*

This is implied by the following result:

Lemma. *Every incompressible, ∂-incompressible annulus A in an I-bundle N over a (possibly non-compact) surface F (without boundary) is isotopic to a vertical annulus - the inverse image of a loop in F.*

Proof of Lemma: The argument is very similar to the proof of the analogous result for incompressible surfaces in Seifert-fibered spaces. First, let p be the projection from N to F, and replace N with p^{-1}(a compact neighborhood of p(A)), so that N is now compact (although the base F now probably has boundary). Now choose an essential arc (or an esential loop to start with, if the base has no boundary), and consider its inverse image under p. Because our annulus A is incompressible and ∂-incompressible, and N is irreducible, we can, using standard innermost loop and outermost arc arguments, isotope A so that it meets the splitting-rectangle over this arc (or annulus/Möbius band, if we began with a loop) only in 'essential' arcs, i.e., ones which run from one horizontal boundary component of the rectangle to the other, and, by a further isotopy, we can assume these arcs are in fact I-fibers of the bundle N (see Figure 1).

But these arcs are also non-trivial in A; otherwise, an I-fiber of N can be homotoped, rel endpoojts, to the horizontal boundary of N; otherwise, the base or our I-bundle is $\mathbb{S}^2$ or $\mathbb{R}P^2$, which allow for no incompressible annuli. But projecting to the base F, this gives a null-homotopic loop in the base, which would lift (thinking of ∂N as a 2-fold covering of F) to a closed loop in ∂N, not an arc, a contradiction. Therefore, they cut A into rectangles. Continuing inductively, splitting N open along these vertical splitting-rectangles, since F can be cut open along a finite number of arcs to a disk, we will arrive at a 3-ball $\mathbb{B}^3$, containing a collection of rectangles meeting the 'vertical' part of $\partial\mathbb{B}^3$ in vertical arcs. These annulus-rectangles can now be easily isotoped to be vertical (see Figure 2) - the horizontal parts of the boundary can first be isotoped to lie directly above one another, and then the rectangles can be isotoped to the obvious vertical disk spanning this boundary. A formal argument would first isotope the annulus-rectangles off of these vertical disks, by an innermost loop argument, and then use the Shönflies Theorem to conclude that the unions of the vertical disks and our annulus-rectangles are a collection of spheres, bounding balls - the balls allow us to isotope the annulus-rectangles to the vertical ones. By gluing N back together again along the splitting-rectangles, our annulus-rectangles give a vertical annulus isotopic to A.

Figure 1 **Figure 2**

Proof of Proposition: One direction of the proposition is clear. M$|\mathcal{L}$ can be built from M\int(N(B)) by gluing on the components of N(B)$|\mathcal{L}$. Each of the components of N(B)$|\mathcal{L}$ are I-bundles, from the foliation of N(B) by intervals. If all of the components of M\int(N(B)) are I-bundles, then, since we can assume that the I-bundle structures along the vertical boundary of M\int (N(B))=∂_vN(B) is the same as the I-bundle structure on the vertical boundary of N(B)$|\mathcal{L}$, the two glue together to give an I-bundle structure to M$|\mathcal{L}$.

For the other (more important) direction, suppose that M_0 = M$|\mathcal{L}$ consists entirely of I-bundles, and consider the annuli ∂_vN(B)$\subseteq M_0$. These annuli split M_0 into M\int (N(B)) and N(B)$|\mathcal{L}$, and are (almost) incompressible and ∂-incompressible in each piece. They are usually essential in M\int (N(B)) by the essentiality of B; a ∂-compressing disk for one of the annuli would give a monogon for B, a contradiction. A compressing disk for one of the annuli, on the other hand, could be pushed up off of ∂_vN(B) to give a compressing disk for ∂_hN(B). Therefore, these ∂_h components must be disks, so the component of M\int (N(B)) is a 2-disk$\times$I, hence a product. We can therefore 'ignore' these pieces, and absorb them into N(B)$|\mathcal{L}$, i.e., we will actually consider the collection of ∂_v-components which do not bound 2-disks$\times$ I.

In the other direction, the inclusion of the base of the I-bundle $M_1 = N(B)|\mathcal{L}$ into the I-bundle (as the set of midpoints of each fiber) is a homotopy equivalence, and the projection of each component of $\partial_v N(B)$ into the base is π_1-injective, since no component of the base is a disk, by essentiality of B - the disk would give a disk of contact for B. Therefore, each annulus π_1-injects into M_1. They are also ∂-injective, since the arc in the ∂I-bundle $M_1 \cap \mathcal{L}$ coming from a ∂-compressing disk would project to the base of the I-bundle as a null-homotopic arc, rel boundary; so since the ∂I-bundle is a 2-fold covering of the base, the null-homotopy lifts to a null-homotopy in $M_1 \cap \mathcal{L}$, rel boundary. But this implies that the endpoints of our arc are in the same ∂-component of our $\partial_v N(B)$ component, a contradiction.

Being incompressible and ∂-incompressible in both directions, they are therefore incompressible and ∂-incompressible in M_0.

By the lemma, all of these annuli can be isotoped to be vertical in $M_0 = M|\mathcal{L}$. Consequently, after the isotopy, each component of $M_0|\partial N(B)$ is saturated by I-fibers of M_0, i.e., is an I-bundle. Consequently, $M \backslash \mathrm{int}(N(B))$, which consists of components of $M_0|\partial_v N(B)$, is a collection of I-bundles.

Corollary. *If $\mathcal{L}$ is an essential lamination in a small Seifert-fibered space, carried with full support by the essential branched surface B, then $M \backslash N(B)$ consists of I-bundles, with $\partial_h N(B)$ corresponding to the associated ∂I-bundles.*

§2
Genuine laminations in 2-bridge knot surgeries

In [De1], Delman constructs essential laminations, in the exterior M(K) of any non-torus 2-bridge knot K, which miss ∂M(K), and which remain essential under all non-trivial Dehn surgeries along K. These laminations have the property that each is carried with full support by a branched surface B = B(K) s.t. the component N of $M(K) \backslash \mathrm{int}(N(B))$ containing ∂M(K) is a 2-torus crossed with I, and the annuli of $\partial_v N(B)$ meeting this component are parallel to the meridian circle of ∂M(K) (see Figure 3). Further, there are at least two such annuli, and, in most cases, there are at least four.

Figure 3

With these facts, we can see that, after non-integer Dehn surgery (if there are two annuli), or, in most cases (when there are at least four annuli), after non-trivial Dehn surgery, the resulting essential lamination is in fact genuine. After Dehn surgery the component N of $M(K) \backslash \mathrm{int}(N(B))$ has been filled in to a solid torus, and the boundary of the meridian disk of this solid torus has intersection number |q| with the core of each of the annuli of $\partial_v N(B)$ in the boundary of the solid torus (see Figure 3). If the lamination is not genuine, then this solid torus has the structure of a product (annulus)×I, with $\partial_v N(B) = \partial(\text{annulus}) \times I$. Consequently, the meridian disk of the solid torus crosses $\partial_v N(B)$ exactly twice.

But under our assumptions above, this meridian disk in fact crosses $\partial_v N(B)$ at least four times; either because there are two annuli crossed by the meridian disk $|q| \geq 2$ times each, or because there are at least four annuli crossed by the meridian disk at least $|q| \geq 1$ times each. Therefore, the essential lamination Delman constructs is genuine, after all non-integer (and usually, non-trivial) Dehn surgeries. Combining this with the result of the previous section, gives us our main theorem.

Remark. In [De2], Delman extends his constructions for 2-bridge knots to other classes of knots which can be built from rational tangles, including most pretzel and Montesinos knots. Since these laminations exhibit the same phenomenon of two, and often four, meridianal annuli, the same conclusions presented here hold for these knots, as well.

Remark. By applying this approach to other laminations in 2-bridge knot exteriors, one can extend these results to show [BW] that every manifold obtained by non-trivial surgery on a non-torus, non-twist 2-bridge knot contains a genuine essential lamination.

§3
The future

New constructions of essential laminations in knot complements continue to be found (see, e.g., [Ro],[Mo]), most of which remain genuine under Dehn surgery, and so giving similar results to those found here. But essential laminations in small Seifert-fibered spaces also have more structure than we have exploited here, leaving open the possibility for even further improvements.

An essential lamination $\mathcal{L}$ in a small Seifert-fibered space M contains a horizontal sublamination, and, if $\mathcal{L}$ does not contain a torus leaf, the results of [Br2] imply that the entire lamination can be made horizontal, hence transverse to the circle fibering of M. If we fill in the product complements of $\mathcal{L}$ with product foliations, and lift the resulting foliation to a covering of M corresponding to a regular circle fiber, we get a foliation transverse to the product circle fibering of an open solid torus. The leaves [GK2] therefore look like open meridianal disks, and so if we lift to the universal covering, we get a foliation by planes of $\mathbb{R}^3$ with space of leaves (i.e., the quotient of $\mathbb{R}^3$ obtained by crushing each leaf to a point) equal to $\mathbb{R}$. Gabai [Ga1] has labelled such foliations (in the base 3-manifold) 'tight'; Fenley [Fe1] calls them '$\mathbb{R}$-covered'..

Therefore, essential laminations in small Seifert-fibered spaces complete to foliations which are tight. This gives us additional information to restrict the possible Dehn surgeries yielding small Seifert-fibered spaces, which we can use in several ways.

First, one could show if one has an essential lamination missing a knot, which does not become genuine under Dehn filling, usually does not become <u>tight</u>, as well. This would amount, basically to showing that there are three lifts of leaves to the universal covering (which are proper planes, by [GO]) for which no one separates the other two - that would contradict space of leaves $\mathbb{R}$, since in $\mathbb{R}$, given three points one separates the other two. In fact, this property is equivalent to having space of leaves $\mathbb{R}$. This suggests the

Question 1. Can the property of having space of leaves $\mathbb{R}$ in the universal covering be detected in the original 3-manifold?

Gabai has pointed out that a finite-depth foliation of depth > 0 (which is not just a 'perturbation' of a depth-0 foliation) <u>never</u> lifts to one with space of leaves

$\mathbb{R}$; there are lifts of the compact leaf which satisfy the condition described above. Since Roberts' constructions [Ro] are variations of Gabai's constructions of depth-1 foliations in alternating knot complements [Ga2], it might be reasonable to expect that those foliations do not have space of leaves $\mathbb{R}$, in almost all cases. This would no doubt also be closely related to the above question.

If a (taut) foliation contains a genuine sublamination, then the foliation certainly cannot be tight (lifts of leaves of the sublamination would fulfill the condition described above - the lift of its non-I-bundle component would not be an I-bundle in the universal covering). Most taut foliations $\mathcal{F}$ contain only one minimal sublamination; for example, if the foliated 3-manifold M is non-Haken [Br3]. If this sublamination is not genuine, then the foliation contains no genuine sublaminations (by essentially the argument in section 1 of [Br1]). Unfortunately, this is not enough to imply that the foliation is tight - Bonatti and Langevin [BL] have found an example of a foliation in a graph manifold in M, all of whose leaves are dense in M, which is not tight. More recently, Fenley [Fe2] has found similar examples among (Haken) hyperbolic 3-manifolds. By [Br3], we have to pass to an infinite covering of M before the lifted foliation would have more than one minimal sublamination, hence any chance of having a genuine sublamination, making it impossible to try a 'virtual' solution along these lines.

We can also try to argue that if an essential lamination, in the exterior $X(K)$ of a knot K does become tight under Dehn surgery, then other Dehn surgeries sufficiently far away cannot make it tight. The second Dehn surgery can be thought of as a Dehn surgery (on essentially the same loop) in the Dehn-surgered manifold that has the tight foliation, along a loop missing the original lamination. This surgery should have the effect of seeming to 'twist up' the leaves of the lamination, possibly destroying the property of having space of leaves $\mathbb{R}$.

Question 2. Can Dehn surgery on a loop missing an essential lamination qualitatively change the lift of the lamination to the universal covering? What about for the motivating example - the core of a solid torus component?

It is important to point out that Dehn surgery need not change the structure of the lift. One can obtain small Seifert-fibered spaces by more than one Dehn surgery on a 2-bridge knot, most notably, by six distinct surgeries on the figure 8 knot [Th]. One can also show [BW] that three Dehn surgeries on a twist (hence 2-bridge) knot will yield small Seifert-fibered spaces. The real problem, for such examples, then, is to determine whether or not surgeries far away from a small Seifert one can also yield small Seifert-fibered spaces. Because of the figure-8 knot examples, 'far away' must mean distance (= geometric intersection number) at least 6, in general; but that is probably largely because of the symmetry of this knot.

Finally, there are questions related to detecting small Seifert-fibered spaces, rather than the reverse. Are small Seifert-fibered spaces the only 3-manifolds which only admit tight foliations? Gabai [Ga1] has asked if laminar manifolds always admit tight foliations or laminations - what we are asking is when do they also admit something else? This would be one way of answering: if a manifold M does admit a tight foliation, can you tell whether or not it is in fact a (small) Seifert-fibered space? Another possible answer might come from the fact that $\pi_1(\mathrm{M})$ acts on the space of leaves $\mathbb{R}$, by its action on the universal covering of M; must this action be qualitatively different depending on whether or not M is Seifert-fibered?

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