

Essential Laminations in Non-Haken 3-manifolds

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In this paper we show that an essential lamination \mathcal{L} in a non-Haken 3-manifold M is ‘tightly wrapped’, in the sense that any two leaves of \mathcal{L} have intersecting closures; \mathcal{L} therefore contains a unique minimal sublamination. We also show that these properties are inherited by any lift of \mathcal{L} to a finite cover of M .

Essential laminations are natural generalizations of incompressible surfaces in a 3-manifold, and give topologists an object which can be found in many more 3-manifolds than an incompressible surface can (see, e.g., [De] or [Na]). Yet they retain enough properties in common with incompressible surfaces so that they can be used to prove some of the powerful results about 3-manifolds that incompressible surfaces have been used to prove (see, e.g., [G-O] or [G-K]).

This note was motivated by asking the question ‘What more can be said about an essential lamination if we know that the 3-manifold M containing it is non-Haken, i.e., M does not contain a (2-sided) incompressible surface?’ What we find is the following ‘structure theorem’:

Theorem 1: Let \mathcal{L} be an essential lamination in a compact, non-Haken 3-manifold. Then \mathcal{L} contains a unique non-empty minimal sublamination \mathcal{L}_0 , i.e., a lamination $\mathcal{L}_0 \subseteq \mathcal{L}$

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with the property that $\mathcal{L}_0 \subseteq \overline{\mathcal{L}}$ for every leaf L of \mathcal{L} . \mathcal{L}_0 can therefore be defined (a posteriori) as $\mathcal{L}_0 = \bigcap_{L \subseteq \mathcal{L}} \overline{L} \neq \emptyset$.

Since a codimension-1 foliation without Reeb components is an example of an essential lamination, we have as an immediate consequence:

Corollary 2: A codimension-1 foliation \mathcal{F} without Reeb components in a compact non-Haken 3-manifold M contains a unique minimal set.

This corollary is of independent interest; it is also an interesting example of a foliation-theoretic result which (seemingly) requires a lamination-theoretic proof. Previously (see [La] or [Co]) it was known that \mathcal{F} has only a finite number of distinct minimal sets (that result does apply much more generally, however).

Thus (in a sense), in a non-Haken 3-manifold essential laminations must be ‘tightly-wrapped’. This restriction on their structure adds a potentially useful new tool to the further study of essential laminations. In particular, since Haken manifolds are already fairly well-understood, this additional structure appears in exactly those 3-manifolds in which essential laminations will be the most useful: those which do not already contain an incompressible surface. It can therefore turn what might be perceived as a liability (the lack of a compact leaf, to which more classical techniques could apply) into an asset (extra information about the ‘shape’ of the lamination); see, e.g., [Br1]. This usefulness will have its limitations, however: in the second section we demonstrate its lack of application to finding Haken finite coverings of non-Haken 3-manifolds, by showing that the same tightness property is inherited by any lift of \mathcal{L} to a finite cover of M .

1. Proof of Theorem 1:

The reader is referred to [G-O] for the basic concepts regarding essential laminations. We will assume throughout that the 3-manifold M in question is compact (hence closed, because M is non-Haken), connected, and orientable.

Proposition 3: If M is an irreducible 3-manifold with non-empty boundary, then either M contains a closed (2-sided) incompressible surface or M is a handlebody.

Proof: Choose a ∂ -component F of M . If it is incompressible, we are done. Otherwise there is a compressing disk D for F , and we begin building a compression body (see [Bo]) for M along F by writing $M = M_0 \cup C_0$, where $C_0 = N(F \cup D)$ (i.e., it is $N(F)$ with a 2-handle attached), and M_0 is the rest of M . ∂M_0 has boundary component $F_0 = \partial N(F \cup D) \setminus F$, and we again ask if this is incompressible in M_0 . Continuing inductively, if at any stage F_i is incompressible in M_i , then it is incompressible in M , because F_i is also incompressible in C_i (turning it upside-down (i.e., turning its handlebody structure upside-down), C_i is basically $N(F_i)$ with a bunch of 1-handles attached, so $F_i \pi_1$ -injects). Otherwise, we can decompose F down to a collection of sphere ∂ -components of some M_n . It is easy to see that M_n is irreducible (inductively), and so M_n then consists of a union of 3-balls = 0-handles; Turning C_n upside-down then demonstrates that M is a union of 0-handles with 1-handles attached, i.e., is a handlebody. ■

Lemma 4: Every lamination \mathcal{L} contains a minimal sublamination, i.e., a non-empty lamination $\mathcal{L}_0 \subseteq \mathcal{L}$ which contains no proper (non-empty) sublamination.

Proof: Look at the collection of non-empty sublaminations of \mathcal{L} , ordered by inclusion. Because M is compact, any sequence $\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \dots$ has a lower bound $\bigcap_i \mathcal{L}_i \neq \emptyset$ (any

finite intersection is non-empty, so the full intersection is; it is a lamination because any leaf of \mathcal{L} which has a point in the intersection is entirely contained in each \mathcal{L}_i , so is entirely contained in their intersection). Therefore, by Zorn's Lemma, this collection contains minimal elements, i.e., non-empty sublaminations properly containing no others. ■

Lemma 5: If \mathcal{L}' is an essential lamination obtained from the essential lamination \mathcal{L} by splitting \mathcal{L} open (see [G-O]) along some finite collection of leaves, and if \mathcal{L}' contains a unique minimal sublamination, then \mathcal{L} contains a unique minimal sublamination.

Proof: From the definition of splitting it follows that there is a continuous surjection $\pi : M \rightarrow M$ with $\pi(\mathcal{L}') = \mathcal{L}$ which carries leaves of \mathcal{L}' onto leaves of \mathcal{L} (just crush the I-fibers of the splitting regions to points).

If \mathcal{L}_0 is a non-empty minimal sublamination of \mathcal{L} , then $\mathcal{L}'_0 = \pi^{-1}(\mathcal{L}_0) \cap \mathcal{L}'$ is a sublamination of \mathcal{L}' . By assumption, \mathcal{L}' contains a unique minimal sublamination C' , so because there is a minimal sublamination of \mathcal{L}' in \mathcal{L}'_0 (there is a minimal sublamination for \mathcal{L}'_0 , thought of on its own, which must then be minimal for \mathcal{L}' , as well), it follows that $C' \subseteq \mathcal{L}'_0$.

But then $\pi(C') \subseteq \pi(\mathcal{L}'_0) = \mathcal{L}_0$ is closed in M (C' is closed hence compact in M , so $\pi(C')$ is compact hence closed in M) and saturated (leaves go to leaves under π), so is a non-empty sublamination of \mathcal{L} contained in \mathcal{L}_0 ; therefore, $\mathcal{L}_0 = \pi(C')$ by the minimality of \mathcal{L}_0 . Therefore every minimal sublamination of \mathcal{L} is equal to $\pi(C')$, i.e., there is only one. ■

Now let $\mathcal{L} \subseteq M$ be an essential lamination carried with full support by an essential branched surface $B \subseteq M$, with M compact.

Proposition 6: If M is non-Haken, then $M \setminus \text{int}(N(B))$ consists of handlebodies; in particular, B is connected.

Proof: By the above proposition, if some component M_0 is not a handlebody, then it contains a closed incompressible surface $F \subseteq M_0$. The branched surface BUF then carries the lamination $\mathcal{L}UF$ with full support, and it is easy to verify that BUF is essential in M . The surface F misses the branch locus of BUF , so BUF has no disks of contact or monogons because B doesn't; $\partial_h N(BUF)$ is incompressible by the choice of F , and has no disk or sphere components because F is not a disk or sphere; and $M \setminus N(BUF)$ is irreducible, again by choice of F . Finally, BUF has no Reeb branched surface: since B doesn't have one, any such would have to include F . But since any lamination carried by F must contain a surface homeomorphic to F (the first leaf you meet falling in along an I-fiber must be a 1-fold cover of a component of $\partial N(F) = F \cup F$), F must be a torus, and the Reeb component must consist of a surface parallel to F together with planar leaves limiting on it from parts of B . But this is impossible, because this would require branch curves along F , which do not exist.

Therefore by [G-O] every leaf of $\mathcal{L}UF$, and F in particular, is π_1 -injective in M , making M Haken (it is already irreducible by [G-O]). But this contradicts our hypothesis, so every component M_0 is a handlebody. It follows that B is connected; if it weren't, then some component of $M \setminus \text{int}(N(B))$ would have more than one ∂ -component. ■

Essential laminations carried by branched surfaces with the above property are studied in [H-O], where they are called 'full'; this result therefore says that every essential lamination in a non-Haken 3-manifold is full.

Proposition 7: M and \mathcal{L} as above. Then for any two leaves L_1, L_2 of \mathcal{L} , the intersection of their closures, $\overline{L_1} \cap \overline{L_2}$ is non-empty.

Proof: Consider the essential lamination $\mathcal{L}_0 = \overline{L_1} \cup \overline{L_2}$, carried (see [G-O]) with full support by some essential branched surface B_0 . Split \mathcal{L}_0 open along L_1 and L_2 ; call the resulting lamination \mathcal{L}' . \mathcal{L}' is still carried by B_0 , and there is a canonical projection $\pi: M \rightarrow M$ with $\pi(\mathcal{L}') = \mathcal{L}_0$ which takes an I-fiber of B_0 to itself. Let A_k , for $k=1,2$, be the inverse image, in \mathcal{L}' , of the leaf L_k , under π . Each consists of one or two leaves, bounding an I-bundle component of $M|\mathcal{L}'$, and each leaf maps onto L_k under π . The essential lamination \mathcal{L}' meets the I-fibers of B_0 in nowhere-dense sets; this is because $L_1 \cup L_2$ is dense in \mathcal{L}_0 , and (in an I-fiber) these points have been replaced by intervals meeting \mathcal{L}' only in their endpoints.

By [G-O], the branched surface B_0 is infinitely splittable to \mathcal{L}' ; there is a sequence B_0, B_1, B_2, \dots of essential branched surfaces carrying \mathcal{L}' (with full support), with $\mathcal{L}' \subseteq N(B_i) \subseteq N(B_{i-1})$, and $\cap N(B_i) = \mathcal{L}'$. Applying Proposition 6 to B_i , we can then conclude that it is connected. Since for each of the leaves A_k , $k=1,2$, the support (in B_i) of the sublamination $\overline{A_k}$ is a (closed) sub-branched surface of B_i , it follows that these supports are not disjoint (otherwise they exhibit B_i as the union of two disjoint closed subsets). So there is an I-fiber α_i of $N(B_i)$ which meets both $\overline{A_1}$ and $\overline{A_2}$. In particular, α_i meets A_1 and A_2 , in points x_i and y_i , respectively.

Consider the set of points $x_i \subseteq A_1$; this sequence has a limit point x in $L \subseteq \overline{A_1}$. But because \mathcal{L}' met transverse arcs in nowhere-dense sets, it follows that the I-fibers of the

$N(B_i)$ must be becoming uniformly short (the I-fibers of the $N(B_i)$ are nested in the I-fibers of $N(B_{i-1})$, so their lengths must tend to zero, because \mathcal{L}' contains no transverse arcs). Therefore, the distance between x_i and y_i must be tending to zero; but then since the sequence x_i tends to x , it follows that the sequence y_i also tends to x . But the y_i are all in A_2 , and so any limit point they have lies in $\overline{A_2}$, and therefore x lies in $\overline{A_2}$, i.e., $x \in \overline{A_1} \cap \overline{A_2}$, which is therefore non-empty. But because $\pi(A_k)=L_k$ (so $A_k \subseteq \pi^{-1}(L_k)$), it then follows that $\emptyset \neq \overline{A_1} \cap \overline{A_2} \subseteq \pi^{-1}(\overline{L_1}) \cap (\pi^{-1}(\overline{L_2})) = \pi^{-1}(\overline{L_1} \cap \overline{L_2})$, so $\overline{L_1} \cap \overline{L_2} \neq \emptyset$ in \mathcal{L}_0 . ■

This now allows us to finish the proofs of the theorem and corollary. First, to unify them, if \mathcal{L} is actually a foliation \mathcal{F} without Reeb components, split it open along a (finite) collection of leaves to make it an essential lamination \mathcal{L} (i.e., so that it is carried by a branched surface). Now by Proposition 7, the closure of any two leaves intersect. Let \mathcal{L}_0 be a minimal sublamination for \mathcal{L} , and write it as $\mathcal{L}_0 = \overline{L_0}$ for L_0 a leaf of \mathcal{L}_0 ($\overline{L_0}$ is a sublamination of \mathcal{L}_0 , and so equals \mathcal{L}_0 by minimality). Then for any leaf L of \mathcal{L} , $\overline{L} \cap \overline{L_0} = \overline{L} \cap \mathcal{L}_0$ is a non-empty sublamination of \mathcal{L}_0 (it is easy to see that it is closed and saturated), and therefore by minimality it equals \mathcal{L}_0 , i.e., $\mathcal{L}_0 \subseteq \overline{L}$ for every leaf L of \mathcal{L} . But if $\mathcal{L}_1 = \overline{L_1}$ is any other minimal sublamination for \mathcal{L} , then the fact that $\overline{L_0} \cap \overline{L_1}$ is non-empty implies that $\mathcal{L}_0 = \overline{L_0} \cap \overline{L_1} = \mathcal{L}_1$, so there is only one non-empty minimal sublamination. Therefore, by Lemma 5, our original lamination (or foliation) contains a unique minimal sublamination.

Finally, we finish with a slight improvement on the result $\mathcal{L}_0 \subseteq \overline{L}$:

Corollary 8: \mathcal{L} , M as above, with unique minimal sublamination \mathcal{L}_0 . Let L be a leaf of \mathcal{L} having an end ϵ (see [Ni]). Then the limit set of the end, $\lim_\epsilon(L)$, contains \mathcal{L}_0 .

Proof: $\lim_{\epsilon}(\mathcal{L})$ is a closed, non-empty, saturated subset of \mathcal{L} , i.e., a sublamination. It therefore contains \mathcal{L}_0 . ■

In other words, not only does every leaf of \mathcal{L} limit on \mathcal{L}_0 , but every end of every leaf does, as well.

2. Haken coverings of non-Haken 3-manifolds

The structure theorem above shows that the leaves of essential laminations in non-Haken 3-manifolds behave in a qualitatively different fashion, in general, from those of essential laminations in Haken manifolds. It therefore gives, in principle, a way to distinguish Haken manifolds from non-Haken ones: if a manifold contains an essential lamination which fails to have the property stated in the theorem, then the manifold must be Haken.

This distinction is potentially useful in determining when a laminar 3-manifold is virtually Haken. If M is non-Haken and contains an essential lamination \mathcal{L} , which therefore has the property that the closures of any two leaves intersect, and has a finite covering $\pi:\tilde{M}\rightarrow M$ such that the inverse image \mathcal{L}' of \mathcal{L} , which is an essential lamination in \tilde{M} , has two leaves with disjoint closures, then \tilde{M} must be Haken, and so M is virtually Haken. This is a nice image; however, that is all it is:

Theorem 9: If $\pi:\tilde{M}\rightarrow M$ is a finite covering, with M non-Haken and $\mathcal{L}\subseteq M$ an essential lamination with $\mathcal{L}_0\subseteq\mathcal{L}$ its unique minimal sublamination, then $\mathcal{L}'_0=\pi^{-1}(\mathcal{L}_0)\subseteq\pi^{-1}(\mathcal{L})=\mathcal{L}'$ is the unique minimal sublamination of \mathcal{L}' ; consequently, any two leaves of \mathcal{L}' have intersecting closures.

This theorem therefore says that an essential lamination in a non-Haken 3-manifold is *really* tightly wrapped; it can't be unwrapped by passing to finite covers of M .

The only ingredient of the proof which we do not already have is the following result:

Proposition 10: If π , \tilde{M} , M , \mathcal{L}_0 , and \mathcal{L}'_0 are as above, and if L'_0 , L' are leaves of \mathcal{L}'_0 with $\overline{L'_0} \subseteq \overline{L'}$, then $\overline{L'_0} = \overline{L'}$.

Proof: Suppose that $\overline{L'_0} \neq \overline{L'}$, so that $\overline{L'_0}$ is properly contained in $\overline{L'}$; in particular, $L' \cap \overline{L'_0} = \emptyset$. We will show that $\pi(\overline{L'_0}) \neq \mathcal{L}_0$, which is a contradiction, because $\pi|_{\overline{L'_0}} : L'_0 \rightarrow L_0$, where L_0 is the leaf of \mathcal{L}_0 which L'_0 maps to, is a covering map, hence onto, so $\mathcal{L}_0 = \overline{L_0} = \overline{\pi(\overline{L'_0})} \subseteq \pi(\overline{L'_0}) \subseteq \mathcal{L}_0$.

To do this, let $N(B)$ be a fibered neighborhood of a branched surface carrying \mathcal{L}_0 , and let $B' = \pi^{-1}(B)$ and $N(B') = \pi^{-1}(N(B))$ (with I-fibers being carried to I-fibers under π). Pick a fiber X of $N(B)$, and consider $\pi^{-1}(X) = X_1 \cup \dots \cup X_n$. Pick a point $x \in \mathcal{L}_0 \cap X$, and let $\{x_1, \dots, x_k\} = \overline{L'_0} \cap \pi^{-1}(x)$, with $x_i \in X_i$, for all i . Because $\overline{L'_0}$ is a closed set, for each $x_i \in \mathcal{L}'_0 \setminus \overline{L'_0}$ (i.e., each $i = k+1, \dots, n$) there is an open neighborhood \mathcal{O}_i of x_i in X_i which misses $\overline{L'_0}$.

Consider $\mathcal{O} = \pi(\mathcal{O}_1) \cap \dots \cap \pi(\mathcal{O}_k) \subseteq X$. Because π maps the fibers X_i homeomorphically to X , this is an open subset of X containing x . Consider $\pi^{-1}(\mathcal{O}) \subseteq \pi^{-1}(X)$; note that $\pi^{-1}(\mathcal{O}) \cap X_i \subseteq \mathcal{O}_i$ for $k+1 \leq i \leq n$.

Now look at $\pi^{-1}(\mathcal{O}) \cap X_k = \mathcal{O}'$ (so $\pi(\mathcal{O}') \subseteq \mathcal{O}$). This is an open neighborhood of x_k in X_k . Because $\overline{L'}$ contains $(\overline{L'_0}$ and hence) x_k , there are points of L' in X_k which pass arbitrarily close to x_k and hence are contained in \mathcal{O}' . Choose one; call it y'_k . Then $y'_k \notin \overline{L'_0}$, and $\pi^{-1}(\pi(y'_k)) \cap X_i = \pi^{-1}(y') \cap X_i \in \mathcal{O}_i$ are not in $\overline{L'_0}$ for $k+1 \leq i \leq n$. So by choosing a different point y' of \mathcal{O} we have increased the number of points, in the inverse image of a point of

$\mathcal{L}_0 \cap X$, which are not in $\overline{L'_0}$. Continuing, we can therefore find a point y of $\mathcal{L}_0 \cap X$ (in fact, in $\mathcal{L}_0 \cap \mathcal{O}$) with $|\pi^{-1}(y) \cap (\mathcal{L}'_0 \setminus L'_0)| = n$, i.e., $\pi^{-1}(y) \subseteq \mathcal{L}'_0 \setminus L'_0$. But this means that $y \notin \pi(\overline{L'_0})$, a contradiction. So $\overline{L'_0} = \overline{L'}$. ■

The above result is a special case of a more general result on the structure of pullbacks of laminations under non-zero degree maps; see [Br1],[Br2].

Proof of Theorem 9: For technical reasons, we must first (as in Proposition 7) split \mathcal{L}_0 open along a leaf, to insure that \mathcal{L}_0 (and hence \mathcal{L}'_0) meets I-fibers of some (hence every) branched surface neighborhood in nowhere-dense sets. We will first prove Theorem 9 for this (possibly different) collection of laminations.

Proposition 10 implies that for any pair of leaves L, L' of \mathcal{L}'_0 , either $\overline{L} \cap \overline{L'} = \emptyset$ or $\overline{L} = \overline{L'}$. Consequently, \mathcal{L}'_0 is the union of a finite number of disjoint minimal sublaminations, $\overline{L}_1, \dots, \overline{L}_n$. This is because the closure of any leaf is a minimal sublamination, and every minimal sublamination of \mathcal{L}'_0 maps onto \mathcal{L}_0 under π . Thus for a given leaf L of \mathcal{L}_0 , every minimal sublamination of \mathcal{L}'_0 contains an inverse image leaf of L , of which there are only finitely many.

Claim: $n=1$.

Otherwise, there is a branched surface closely approximating \mathcal{L}'_0 which is not connected. To see this, just choose one whose fibered neighborhood has fibers with length less than $\delta =$ half of the distance between two components of \mathcal{L}'_0 ; this can be done because (as in Proposition 7) \mathcal{L}'_0 meets I-fibers in nowhere-dense sets). But this also means that there is an (essential) branched surface B carrying \mathcal{L}_0 with full support whose inverse image under π is not connected; choose one with fibers of the same short length δ .

But this situation is absurd; B has complement consisting entirely of handlebodies, hence so does $\pi^{-1}(B)$ (the only thing that finitely covers a handlebody is a handlebody), so every component of $\tilde{M} \setminus N(\pi^{-1}(B))$ has connected boundary, so $\pi^{-1}(B)$ is connected.

Therefore \mathcal{L}'_0 is a minimal sublamination of \mathcal{L}' . But then since for every leaf L' of \mathcal{L}' , $\overline{\pi(L')} \subseteq \pi(\overline{L'})$, and since $\mathcal{L}_0 \subseteq \overline{L} = \overline{\pi(L')}$ for $\pi(L') \subseteq L \subseteq \mathcal{L}$ (since $\pi(L') = L$), we have that $\pi(\overline{L'}) \cap \mathcal{L}_0 \neq \emptyset$, so $\overline{L'} \cap \pi^{-1}(\mathcal{L}_0) = \overline{L'} \cap \mathcal{L}'_0 \neq \emptyset$. Therefore $\mathcal{L}'_0 \subseteq \overline{L'}$. So any two leaves of \mathcal{L}' have closures whose intersection contains \mathcal{L}'_0 ; it is therefore the unique minimal sublamination of \mathcal{L}' .

But now if we collapse our split open leaves of \mathcal{L}'_0 back again, Lemma 5 then insures that \mathcal{L}' contains a unique minimal sublamination; in particular, it is the image under collapsing of the minimal sublamination \mathcal{L}'_0 , i.e., it is the original \mathcal{L}'_0 . ■

As a final note, we should point out that not only is this true for $\pi^{-1}(\mathcal{L}) = \mathcal{L}'$, but also for any sublamination \mathcal{L}'_1 of \mathcal{L}' ; the closure of any leaf of \mathcal{L}'_1 is the same whether we think we are in \mathcal{L}'_1 or \mathcal{L}' , so it contains \mathcal{L}'_0 . In particular, any essential branched surface carrying \mathcal{L}'_1 has complement consisting of handlebodies. For otherwise (by the proof of Proposition 6) there would be an incompressible surface in \tilde{M} missing \mathcal{L}'_1 , hence missing \mathcal{L}'_0 , hence contained in the handlebody complement of some essential branched surface carrying \mathcal{L}'_0 , a contradiction.

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1-22

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