Hyperbolic Manifolds

Hilary Term 2000

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Geometry and topology is, more often than not, the study of manifolds. These manifolds come in a variety of different flavours: smooth manifolds, topological manifolds, and so on, and many will have extra structure, like complex manifolds or symplectic manifolds. All of these concepts can be brought together into one overall definition.

A *pseudogroup* on a (topological) manifold X is a set $\mathcal{G}$ of homeomorphisms between open subsets of X satisfying the following conditions:

1. The domains of the elements of $\mathcal{G}$ must cover X.
2. The restriction of any element of $\mathcal{G}$ to any open set in its domain is also in $\mathcal{G}$.
3. The composition of two elements of $\mathcal{G}$, when defined, is also in $\mathcal{G}$.
4. The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
5. The property of being in $\mathcal{G}$ is 'local', that is, if $g: U \rightarrow V$ is a homeomorphism between open sets of X, and U has a cover by open sets U_α such that $g|_{U_\alpha}$ is in $\mathcal{G}$ for each U_α, then g is in $\mathcal{G}$.

For example, the set of all diffeomorphisms between open sets of $\mathbb{R}^n$ forms a pseudogroup.

A *$\mathcal{G}$-manifold* is a Hausdorff topological space M with a countable $\mathcal{G}$-atlas. A *$\mathcal{G}$-atlas* is a collection of $\mathcal{G}$-compatible co-ordinate charts whose domains cover M. A *co-ordinate chart* is a pair (U_i, ϕ_i), where U_i is an open set in M and $\phi_i: U_i \rightarrow X$ is a homeomorphism onto its image. That these are *$\mathcal{G}$-compatible* means that whenever (U_i, ϕ_i) and (U_j, ϕ_j) intersect, the transition map $\phi_i \circ \phi_j^{-1}: \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$ is in the pseudogroup $\mathcal{G}$.

Unless otherwise stated, all manifolds we will consider will be connected, Hausdorff and second countable.

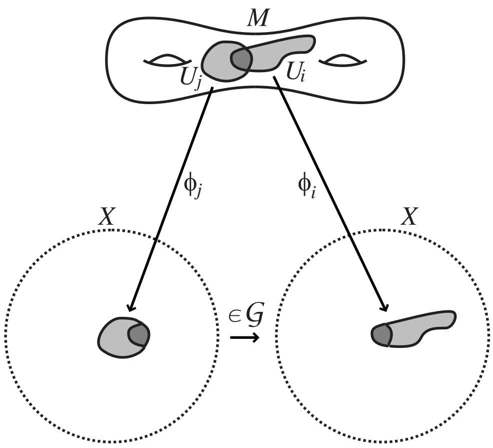


Figure 1.

Examples.

X	Pseudogroup $\mathcal{G}$	$\mathcal{G}$-manifold
$\mathbb{R}^n$	All homeomorphisms between open subsets of $\mathbb{R}^n$	Topological manifold
$\mathbb{R}^n$	All C^r-diffeomorphisms between open subsets of $\mathbb{R}^n$ $(1 \leq r \leq \infty)$	Differentiable manifold (of class C^r)
$\mathbb{C}^n$	All biholomorphic maps between open subsets of $\mathbb{C}^n$	Complex manifold

Other examples. Real analytic manifolds, foliated manifolds, contact manifolds, symplectic manifolds, piecewise linear manifolds.

The above definition of a $\mathcal{G}$-manifold was actually a little ambiguous. When is it possible for two different $\mathcal{G}$-atlases to define the same $\mathcal{G}$-structure? Two $\mathcal{G}$-atlases on a topological space M define the same $\mathcal{G}$-structure if they are *compatible*, which means that their union is also a $\mathcal{G}$-atlas. Compatibility is an equivalence relation (exercise) and hence any $\mathcal{G}$-atlas is contained in a well-defined equivalence class of $\mathcal{G}$-atlases.

Exercise. Let $\mathcal{G}$ be the set of translations of $\mathbb{R}$ restricted to open subsets of $\mathbb{R}$. Show that $\mathcal{G}$ satisfies the first four conditions in the definition of a pseudogroup, but fails the fifth condition. Show that compatibility between $\mathcal{G}$-atlases on S^1 is not an equivalence relation.

Let M be a $\mathcal{G}$-manifold and let $h\colon N \to M$ be a local homeomorphism (that is, each point of N has an open neighbourhood U such that $h|_U$ is an open mapping that is a homeomorphism onto its image). Then we may pull back the $\mathcal{G}$-structure on M to a $\mathcal{G}$-structure on N.

A homeomorphism $h\colon N \to M$ between $\mathcal{G}$-manifolds is a *$\mathcal{G}$-isomorphism* if the pull back $\mathcal{G}$-structure on N is the same as the $\mathcal{G}$-structure it possesses already.

Let $\mathcal{G}_0$ be a collection of homeomorphisms between open subsets of a manifold X. The pseudogroup $\mathcal{G}$ *generated by* $\mathcal{G}_0$ is the intersection of all pseudogroups on X containing $\mathcal{G}_0$. It is the smallest pseudogroup containing $\mathcal{G}_0$.

In certain cases, it is possible to identify the pseudogroup that is generated much more explicitly.

Special case. Let G be a group acting on a manifold X. Let $\mathcal{G}$ be the pseudogroup generated by G. Then $g \in \mathcal{G}$ if and only if the domain of g can be covered by open sets U_α such that $g|_{U_\alpha} = g_\alpha|_{U_\alpha}$ for some $g_\alpha \in G$ (exercise). A $\mathcal{G}$-manifold is also called a (G, X)-*manifold*.

Terminology.

X	G	(G, X)-manifold
$\mathbb{R}^n$	Euclidean isometries	Euclidean manifold
S^n	Spherical isometries	Spherical manifold
$\mathbb{R}^n$	Affine transformations	Affine manifold
$\mathbb{R}^n$	Euclidean similarities	Similarity manifold

In each of these cases, the group G is quite small (much smaller than the full diffeomorphism pseudogroup) and so the resulting (G, X)-structures are quite rigid.

Examples. 1. By taking a single chart, any open subset of $\mathbb{R}^n$ is a (G, X)-manifold for all (G, X).

2. The torus admits a Euclidean structure, with the following charts.

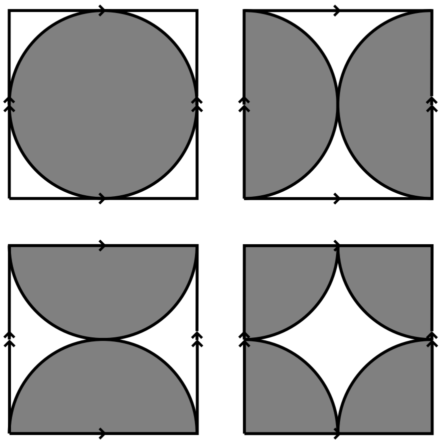


Figure 2.

Another way of constructing this example is as follows. Let M be a manifold and let $\tilde{M}$ be its universal cover, with G the group of covering transformations. Then M inherits a $(G, \tilde{M})$-structure.

The action of a group G on a manifold X is *rigid* if, whenever two elements of G agree on an open set of X, they are the same element of G. Then the pseudogroup generated by such a G is the set of homeomorphisms $h: U \to h(U)$ between open subsets of X such that the restriction of h to any component of U is the restriction of an element of G. The examples above of groups G acting on a manifold X are all rigid. Also, if X is a Riemannian manifold and G is a group of isometries of X, then G acts rigidly. (This is a consequence of Theorem 1.5 of the Introduction to Riemannian Manifolds.)

Euclidean structures are very well understood, as demonstrated by the following result.

Theorem. [Bieberbach] *Every closed Euclidean n-manifold is finitely covered by a torus T^n.*

For example, the only closed surfaces that support Euclidean structures are the torus and the Klein bottle. Spherical structures are even more restrictive.

Theorem. *A closed spherical n-manifold is finitely covered by S^n. In particular, it has finite fundamental group.*

There is a fascinating conjectured converse to this in dimension three.

Conjecture. *Every closed 3-manifold with finite fundamental group admits a spherical structure.*

This implies the famous Poincaré conjecture.

Poincaré Conjecture. *A closed 3-manifold with trivial fundamental group is homeomorphic to S^3.*

In this course we will define and study another model space X. We will define, for each $n \geq 1$, a Riemannian n-manifold $\mathbb{H}^n$, known as *hyperbolic space*. Its isometry group is denoted by $\mathrm{Isom}(\mathbb{H}^n)$. An $(\mathrm{Isom}(\mathbb{H}^n), \mathbb{H}^n)$-manifold is known as a *hyperbolic manifold*. A hyperbolic manifold inherits a Riemannian metric.

It is a theorem from Riemannian geometry that $\mathbb{H}^n$ (respectively, S^n, Euclidean space) is the unique complete simply-connected Riemannian n-manifold with all sectional curvatures being -1 (respectively, one, zero). Hyperbolic manifolds are precisely those Riemannian manifolds in which all sectional curvatures are -1.

Hyperbolic space has a richer isometry group than Euclidean or spherical space, and hence it will be easier to find hyperbolic structures. But still, hyperbolic manifolds are sufficiently rigid to have interesting properties. Here are some sample results about hyperbolic manifolds.

A smooth 3-manifold is *irreducible* if any smoothly embedded 2-sphere bounds a 3-ball. A smooth 3-manifold M is *atoroidal* if any $\mathbb{Z} \oplus \mathbb{Z}$ subgroups of $\pi_1(M)$ is conjugate to $i_*(\pi_1(X))$, where $i \colon X \to M$ is the inclusion of a toral boundary component of M. A compact orientable 3-manifold M is *Haken* if it is irreducible and it contains a compact orientable embedded surface S (other than a 2-sphere) with $\partial S = S \cap \partial M$, such that the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is an injection. Haken 3-manifolds form a large class. In particular, any compact orientable irreducible 3-manifold M with non-empty boundary or with infinite $H_1(M)$ is Haken.

Theorem. [Thurston] *Let M be a closed atoroidal Haken 3-manifold. Then M admits a hyperbolic structure.*

This is a special case of the so-called geometrisation conjecture.

Geometrisation Conjecture. [Thurston] *Any closed irreducible atoroidal 3-manifold admits either a hyperbolic structure or a spherical structure.*

The closed irreducible toroidal 3-manifolds with $\mathbb{Z} \oplus \mathbb{Z}$ subgroups in their fundamental group are known to admit a certain type of 'geometric structure', but the spaces X on which they are modelled have slightly less natural geometries.

The above theorems and conjectures suggest that it may be rather too easy to put a hyperbolic structure structure on a manifold. But in fact this is not the case.

Theorem. [Mostow Rigidity] *Let M and N be closed hyperbolic n-manifolds, with $n > 2$. If $\pi_1(M)$ and $\pi_1(N)$ are isomorphic, then M and N are isomorphic hyperbolic manifolds.*

This is very strong indeed. It says that each of the following implications can be reversed for closed hyperbolic n-manifolds for $n > 2$:

$$\text{Isomorphic} \Rightarrow \text{Isometric} \Rightarrow \text{Diffeomorphic}$$

$$\Rightarrow \text{Homeomorphic} \Rightarrow \text{Homotopy equivalent} \Rightarrow \text{Isomorphic } \pi_1$$

There are lots of geometric invariants of hyperbolic manifolds. For example, they have a well-defined volume. Thus Mostow Rigidity implies that these depend only on the fundamental group of the manifold. In particular, they are topological invariants.

In the case of hyperbolic manifolds, it is those that are complete which are particularly interesting. Mostow's rigidity theorem remains true when the word 'closed' is replaced by 'complete and finite volume'.

Thurston's theorem on the hyperbolisation of closed atoroidal Haken manifolds extends the bounded case as follows.

Theorem. [Thurston] *Let M be a compact orientable irreducible atoroidal 3-manifold, such that ∂M is a non-empty collection of tori. Then either $M - \partial M$ has a complete finite volume hyperbolic structure, or M is homeomorphic to one of the following exceptional cases:*

1. $S^1 \times [0, 1] \times [0, 1]$

2. $S^1 \times S^1 \times [0,1]$

3. the space obtained by gluing the faces of a cube as follows: arrange the six faces into three opposing pairs; glue one pair, by translating one face onto the other; glue another pair, by translating one face onto the other and then rotating through π *about the axis between the two faces. (This is the total space of the unique orientable* I*-bundle over the Klein bottle.)*

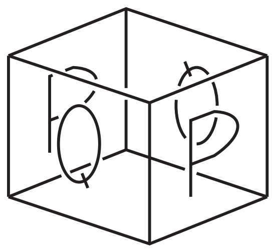


Figure 3.

Example. Let K be a knot in S^3, that is, a smoothly embedded simple closed curve. Let $N(K)$ be an open tubular neighbourhood of K. Then $M = S^3 - N(K)$ is a 3-manifold with boundary a torus, which is compact, orientable and irreducible. Irreducibility is a consequence of the Schoenflies theorem. Note that $M - \partial M$ is homeomorphic to $S^3 - K$. The knots K for which M fails to be atoroidal fall into one of two classes:

1. *torus knots*, which are those that lie on the boundary of a standardly embedded solid torus in S^3, and are not the unknot, and
2. *satellite knots*, which are those that have an embedded π_1-injective torus in their complement that is not boundary-parallel. Such a torus bounds a 'knotted' solid torus in S^3 containing the knot.

So, providing K is neither the unknot, a torus knot nor satellite knot, $S^3 - K$ admits a complete, finite volume hyperbolic structure.

1. Three models for hyperbolic space

Hyperbolic space was discovered by a number of people, including Bolyai, Gauss and Lobachevsky. It has many of the properties of Euclidean space, including:

1. Between any two points in $\mathbb{H}^n$, there is a unique geodesic.

2. For any two points $x, y \in \mathbb{H}^n$, there is an isometry taking x to y.

   However, there is a major difference between $\mathbb{H}^2$ and $\mathbb{R}^2$:

3. If α is a geodesic in $\mathbb{H}^2$ and x is a point not on α, then there are infinitely many geodesics through x which do not meet α.

Remark. We will often confuse a geodesic $\alpha\colon [0,T] \to M$ with its image in M. Thus 'unique geodesic' really means 'unique up to re-parametrisation'.

There are three main 'models' for hyperbolic space, each of which is a Riemannian manifold, any two of which are isometric. Each will be denoted by $\mathbb{H}^n$.

The Poincaré disc model

For each $n \in \mathbb{N}$, let D^n be the open unit ball $\{x \in \mathbb{R}^n \colon d_{Eucl}(x,0) < 1\}$, where $d_{Eucl}(x,0)$ is the Euclidean distance between x and the origin 0 in $\mathbb{R}^n$. Assign a Riemannian metric to D^n by defining the inner product of two vectors v and w in $T_x D^n$ to be

$$\langle v, w\rangle_{D^n} = \langle v, w\rangle_{Eucl} \left(\frac{2}{1 - [d_{Eucl}(x,0)]^2} \right)^2,$$

where $\langle\ ,\ \rangle_{Eucl}$ is the standard Euclidean inner product. This is the Poincaré disc model for $\mathbb{H}^n$.

Remarks. 1. Since $\langle v, w\rangle_{D^n}$ is a multiple of $\langle v, w\rangle_{Eucl}$, the angle between two non-zero vectors in $T_x\mathbb{H}^n$ is just their Euclidean angle.

2. The factor $2/(1 - [d_{Eucl}(x,0)]^2) \to \infty$ as $d_{Eucl}(x,0) \to 1$, so hyperbolic distances get very big as $d_{Eucl}(x,0) \to 1$.

3. The inclusions $D^1 \subset D^2 \subset D^3 \subset ...$ induce inclusions $\mathbb{H}^1 \subset \mathbb{H}^2 \subset \mathbb{H}^3$ which respect the Riemannian metrics.

The unit sphere $\{x \in \mathbb{R}^n : d_{Eucl}(x, 0) = 1\}$ is known as the *sphere at infinity* S^{n-1}_∞. It is *not* part of hyperbolic space. But it is nonetheless useful when studying $\mathbb{H}^n$.

The upper half-space model

This is another way of describing hyperbolic space. Let $U^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Give it a Riemannian metric by defining the inner product of v and w in $T_{(x_1,\ldots,x_n)}U^n$ to be

$$\langle v, w\rangle_{U^n} = \frac{\langle v, w\rangle_{Eucl}}{x_n^2}.$$

Proposition 1.1. *There is a Riemannian isometry between D^n and U^n.*

Proof. Let $\pm e_n = (0, 0, \ldots, \pm 1) \in \mathbb{R}^n$. Consider the map

$$\begin{aligned} \mathbb{R}^n - \{-e_n\} &\xrightarrow{I} \mathbb{R}^n - \{-e_n\} \\ x &\mapsto 2\frac{x + e_n}{[d_{Eucl}(x + e_n, 0)]^2} - e_n. \end{aligned}$$

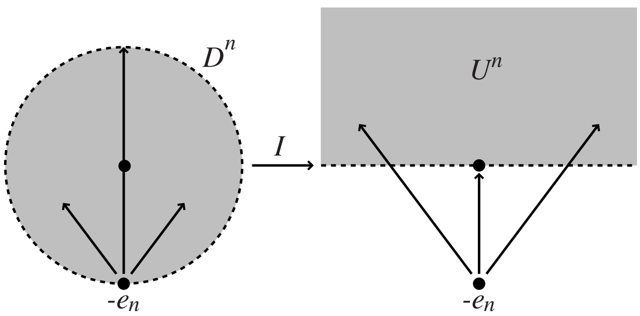


Figure 4.

Let $p_n\colon \mathbb{R}^n \to \mathbb{R}$ be projection onto the n^{th} co-ordinate. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n - \{-e_n\}$. Then

$$I(x) \in U^n \Leftrightarrow p_n(I(x)) > 0$$

$$\Leftrightarrow p_n\left(2\frac{x + e_n}{[d_{Eucl}(x + e_n, 0)]^2} - e_n\right) > 0 \Leftrightarrow p_n\left(2\frac{x + e_n}{[d_{Eucl}(x + e_n, 0)]^2}\right) > 1$$

$$\Leftrightarrow \frac{2(x_n + 1)}{x_1^2 + \ldots + x_{n-1}^2 + (x_n + 1)^2} > 1$$

$$\Leftrightarrow x_1^2 + \ldots + x_n^2 < 1 \Leftrightarrow x \in D^n.$$

So I restricts to a diffeomorphism $D^n \to U^n$.

We now check that it is a Riemannian isometry. Note that $(DI)_x$ acts on $T_x D^n$ by scaling by a factor of $2/[d_{Eucl}(x+e_n, 0)]^2$, then reflecting in the direction of the line joining x to $-e_n$. So,

$$\begin{aligned}
||(DI)_x(v)||_{U^n} &= \frac{||(DI)_x(v)||_{Eucl}}{p_n(I(x))} \\
&= \left(\frac{2||v||_{Eucl}}{[d_{Eucl}(x+e_n, 0)]^2} \right) \left(\frac{2(x_n+1)}{[d_{Eucl}(x+e_n, 0)]^2} - 1 \right)^{-1} \\
&= \frac{2||v||_{Eucl}}{2x_n + 2 - (x_1^2 + \ldots x_{n-1}^2 + (x_n+1)^2)} \\
&= \frac{2||v||_{Eucl}}{1 - [d_{Eucl}(x,0)]^2} \\
&= ||v||_{D^n}.
\end{aligned}$$

So, I is a Riemannian isometry. □

Note. The map I takes $S_\infty^{n-1} - \{-e_n\}$ to $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$. Therefore, we view the 'sphere at infinity' for U^n to be the plane $\partial U^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$, together with a single 'point at infinity', written ∞. In the case $n = 3$, this is the well-known observation that the Riemann sphere is just the complex plane with a single point added.

Note. The isometry $I\colon D^n \to U^n$ is a composition of Euclidean translations and scales, and the map $x \mapsto x/[d_{Eucl}(x, 0)]^2$, which is known as a *Euclidean inversion*.

Lemma 1.2. *I preserves the set $\{$Euclidean planes of dimension $k\} \cup \{$Euclidean spheres of dimension $k\}$.*

Proof. Since Euclidean scales and translations preserve this set, it suffices to show that Euclidean inversion does also. First consider the special case $k = n - 1$. Then, spheres and planes have the form

$$\{x \in \mathbb{R}^n : k_1 \langle x, x\rangle_{Eucl} + k_2 \langle x, a\rangle_{Eucl} + k_3 = 0\},$$

where k_1, k_2 and k_3 are real numbers, not all zero, and a is a vector in $\mathbb{R}^n$, and where these are chosen so that more than one x satisfies the equation. If we invert, this set is sent to the set

$$\begin{aligned}
&\{x \in \mathbb{R}^n : k_1 \frac{\langle x, x\rangle_{Eucl}}{\langle x, x\rangle_{Eucl}^2} + k_2 \frac{\langle x, a\rangle_{Eucl}}{\langle x, x\rangle_{Eucl}} + k_3 = 0\} \\
=&\{x \in \mathbb{R}^n : k_1 + k_2 \langle x, a\rangle_{Eucl} + k_3 \langle x, x\rangle_{Eucl} = 0\},
\end{aligned}$$

which, again is the equation of a sphere or plane in $\mathbb{R}^n$.

Now, consider the general case, where $1 \leq k \leq n-1$. Then a k-dimensional plane or sphere is the intersection of a collection of $(n-1)$-dimensional planes or spheres. This is mapped to the intersection of a collection of $(n-1)$-dimensional planes or spheres, which is an l-dimensional plane or sphere. Since I is a diffeomorphism, it preserves the dimension of submanifolds, and so $l = k$. $\square$

The Klein model

Let $K^n = \{x \in \mathbb{R}^n : d_{Eucl}(x, 0) < 1\}$. Define

$$\begin{aligned}
\phi \colon D^n &\to K^n \\
x &\mapsto x \left(\frac{2 d_{Eucl}(x, 0)}{[d_{Eucl}(x, 0)]^2 + 1} \right).
\end{aligned}$$

Assign K^n the metric that makes ϕ an isometry. This is the Klein model for $\mathbb{H}^n$. Unlike the other two models, angles in K^n do not agree with Euclidean angles. However, we will see that geodesics in K^n are Euclidean geodesics, after re-parametrisation.

2. Some isometries of $\mathbb{H}^n$.

Note that isometries $D^n \to D^n$ are in one-one correspondence with isometries $U^n \to U^n$, by conjugating by $I \colon D^n \to U^n$. Using this, we'll feel free to jump between the two different models we have for hyperbolic space.

Examples. 1. Any linear orthogonal map $h \colon \mathbb{R}^n \to \mathbb{R}^n$ fixing the origin restricts to an isometry $D^n \to D^n$. By considering $I \circ h \circ I^{-1}$ and noting that $I(0) = e_n$, we can find an isometry of U^n fixing e_n, and which realizes any orthogonal map $T_{e_n} U^n \to T_{e_n} U^n$.

2. Consider the map

$$\mathbb{R}^n \xrightarrow{h} \mathbb{R}^n$$

$$x \mapsto \lambda A x + b,$$

where $\lambda \in \mathbb{R}_{>0}$, A is an orthogonal map preserving the e_n-axis and $b \in \mathbb{R}^{n-1} \times \{0\}$. This restricts to a map $U^n \to U^n$ which is a hyperbolic isometry: if $x \in U^n$ and $v \in T_x U^n$, then

$$||(Dh)_x(v)||_{hyp} = ||\lambda v||_{hyp} = \frac{||\lambda v||_{Eucl}}{p_n(\lambda x)} = \frac{||v||_{Eucl}}{p_n(x)} = ||v||_{hyp}.$$

Theorem 2.1. *For any two points x and y in $\mathbb{H}^n$ and any orthogonal map $A: T_x\mathbb{H}^n \to T_y\mathbb{H}^n$, there is an isometry $h: \mathbb{H}^n \to \mathbb{H}^n$ such that $h(x) = y$ and $(Dh)_x = A$. Moreover, h is a composition of isometries as in Examples 1, 2 and 3.*

Proof. Consider x and y in U^n. By using Example 2, we may find isometries f and g such that $f(x) = g(y) = e_n$. Now, $(Dg)_y \circ A \circ (Df^{-1})_{e_n}$ is an orthogonal map $T_{e_n}U^n \to T_{e_n}U^n$ and so is realised by an isometry h fixing e_n, as in Example 1. Therefore, $g^{-1} \circ h \circ f$ is the required isometry. □

Definition. $\mathrm{Isom}(\mathbb{H}^n)$ is the group of isometries of $\mathbb{H}^n$. $\mathrm{Isom}^+(\mathbb{H}^n)$ is the subgroup of orientation-preserving isometries.

Corollary 2.2. *The isometries of Examples 1 and 2 generate* $\mathrm{Isom}(\mathbb{H}^n)$.

Proof. Suppose $h \in \mathrm{Isom}(\mathbb{H}^n)$. Pick $x \in \mathbb{H}^n$. By Theorem 2.1, there is an isometry $g: \mathbb{H}^n \to \mathbb{H}^n$ such that $g(x) = h(x)$ and $(Dg)_x = (Dh)_x$, with g a composition of isometries as in Examples 1, 2 and 3. By Theorem 1.5 of the Introduction of Riemannian manifolds, $h = g$. □

Corollary 2.3. *Any hyperbolic isometry $D^n \to D^n$ (respectively, $U^n \to U^n$)*

1. *extends to a homeomorphism $S^{n-1}_\infty \to S^{n-1}_\infty$ (respectively, $\partial U^n \cup \{\infty\} \to \partial U^n \cup \{\infty\}$),*
2. *preserves*

$$\{\text{Euclidean planes of dimension}\, k\} \cup \{\text{Euclidean spheres of dimension}\, k\},$$

3. *preserves the angles between S^{n-1}_∞ and arcs intersecting S^{n-1}_∞ (respectively, ∂U^n).*

Proof. These are all true for Examples 1 and 2. □

Let 0 be the origin in D^n.

Lemma 3.1. *For any point $x \in D^n - \{0\}$, the unit speed path α running along the Euclidean straight line L through 0 and x is a shortest path from 0 to x in the hyperbolic metric. Hence, it is a geodesic in D^n.*

Proof. Let $\alpha_1\colon [0,T] \to D^n$ be another path from 0 to x in D^n. Our aim is to show that $\text{Length}_{hyp}(\alpha_1) \geq \text{Length}_{hyp}(\alpha)$. We may assume that $\alpha_1^{-1}(0) = 0$. Let α_2 be the path running along L such that

$$d_{Eucl}(\alpha_2(t), 0) = d_{Eucl}(\alpha_1(t), 0)$$

for all $t \in [0,T]$. Then,

$$||\alpha_2'(t)||_{Eucl} \leq ||\alpha_1'(t)||_{Eucl}.$$

Since $||\,||_{Eucl}$ and $||\,||_{hyp}$ differ by a factor which depends only on the Euclidean distance from 0, we have that

$$||\alpha_2'(t)||_{hyp} \leq ||\alpha_1'(t)||_{hyp}.$$

So,

$$\text{Length}_{hyp}(\alpha_2) = \int_0^T ||\alpha_2'(t)||_{hyp}\, dt \leq \int_0^T ||\alpha_1'(t)||_{hyp}\, dt = \text{Length}_{hyp}(\alpha_1),$$

But then $\alpha^{-1} \circ \alpha_2$ is a function $f\colon [0,T] \to [0, \text{Length}_{hyp}(\alpha)]$, such that $|f'(t)| = ||\alpha_2'(t)||_{hyp}$. Then

$$\text{Length}_{hyp}(\alpha_2) = \int_0^T ||\alpha_2'(t)||_{hyp}\, dt = \int_0^T |f'(t)|\, dt \geq$$

$$\int_0^T f'(t)\, dt = f(T) - f(0) = \text{Length}_{hyp}(\alpha).$$

Hence, α is a shortest path from 0 to x. □

Corollary 3.2. *The unit speed geodesic α in Lemma 3.1 is the unique geodesic between 0 and x (up to re-parametrisation).*

Proof. Suppose that α_1 is another geodesic between 0 and x. By Lemma 3.1, α_1 is a Euclidean straight line. Since it goes through x, $\alpha_1'(0)$ is a multiple of $\alpha'(0)$, and so α_1 is a re-parametrisation of α. □

Corollary 3.3. *Between any two distinct points in $\mathbb{H}^n$, there is a unique geodesic.*

Proof. Let x and y be distinct points in D^n. By Theorem 2.1, there is an isometry $h\colon D^n \to D^n$ which takes x to 0. This induces a bijection between geodesics through x and geodesics through 0. By Corollary 3.2, there is a unique geodesic between 0 and $h(y)$. So, there is a unique geodesic between x and y. □

Theorem 3.4. *Geodesics in D^n (respectively, U^n) are precisely the Euclidean straight lines and circles which hit S_∞^{n-1} (respectively, ∂U^n) at right angles.*

Proof. Let α be a path in D^n. Let x be a point on α. By Theorem 2.1, there is an isometry $h\colon D^n \to D^n$ such that $h(x) = 0$. Then

$$\begin{gathered}
\alpha \text{ is a geodesic} \\
\Leftrightarrow h(\alpha) \text{ is a geodesic} \\
\Leftrightarrow h(\alpha) \text{ is a Euclidean straight line through } 0 \\
\Leftrightarrow h(\alpha) \text{ is a Euclidean straight line or circle} \\
\text{through 0 which hits } S_\infty^{n-1} \text{ at right angles} \\
\Leftrightarrow \alpha \text{ is a Euclidean straight line or circle hitting } S_\infty^{n-1} \text{ at right angles.}
\end{gathered}$$

Note that in the above, we used the fact that Euclidean circles through 0 do not hit S_∞^{n-1} at right angles. □

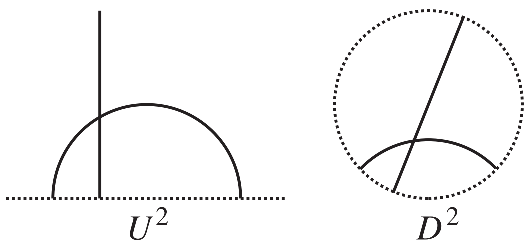
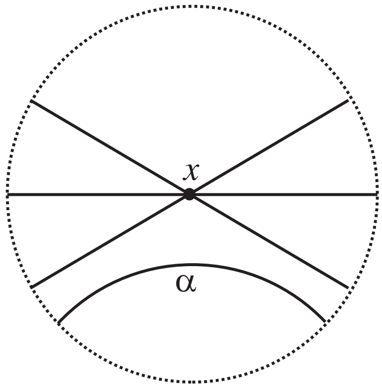


Figure 5.

Corollary 3.5. *If α is a geodesic in $\mathbb{H}^2$ and x is a point not on α, then there are infinitely many distinct geodesics through x which miss α.*



D^2

Figure 6.

Corollary 3.6. *Between any two distinct points x and y in S_∞^{n-1}, there is a unique geodesic.*

Proof. Work with U^n. After an isometry, we may assume that $x = \infty$. So $y \in \partial U^n$. By Theorem 3.4, geodesics through y are Euclidean straight lines and circles meeting y at right-angles. Therefore, the vertical straight line through y is the unique geodesic joining x to y. $\square$

Corollary 3.7. *All geodesics in $\mathbb{H}^n$ are infinitely long in both directions.*

Proof. Let α be a geodesic in U^n. Perform an isometry of U^n so that it passes through e_n and runs to the point at ∞. Re-parametrise α so that the n^{th} coordinate of $\alpha(y)$ is y. Its Euclidean speed is then constant, but its hyperbolic speed is not. Its length between e_n and ∞ is

$$\int_1^\infty ||\alpha'(y)||_{hyp}\, dy = \int_1^\infty 1/y\, dy = [\ln(y)]_1^\infty = \infty. \qquad \square$$

Proposition 3.8. *Geodesics in the Klein model are Euclidean geodesics (up to re-parametrisation).*

Proof. We claim that the function $\phi\colon D^n \to K^n$ is described as follows. Embed D^n in D^{n+1}. At a point x of D^n, let γ be the geodesic perpendicular to D^n through x. Let $(x_1, \dots, x_n, \pm x_{n+1})$ be the endpoints of γ on S_∞^n. Then $\phi(x) = (x_1, \dots, x_n)$. This is a simple calculation in Euclidean geometry.

A codimension one hyperplane in D^n is described by an $(n-1)$-sphere intersecting S_∞^{n-1} orthogonally. It is the intersection of D^n with an n-sphere S.

The vertical projection of $S \cap S^n_\infty$ to D^n is a Euclidean plane. Therefore ϕ maps codimension one hyperplanes in D^n to Euclidean planes. It therefore maps a hyperbolic geodesic, which is the intersection of hyperplanes, to a Euclidean geodesic. □

4. Classification of hyperbolic isometries

Recall the examples of hyperbolic isometries given in §2. The aim now is to show that every hyperbolic isometry is essentially one of these three types.

Definition. Let $h\colon \mathbb{H}^n \to \mathbb{H}^n$ be an isometry. Then h is

(i) *elliptic* if it fixes a point in $\mathbb{H}^n$;

(ii) *parabolic* if it has no fixed point in $\mathbb{H}^n$ and a unique fixed point in S^{n-1}_∞;

(iii) *loxodromic* if it has no fixed point in $\mathbb{H}^n$ and precisely two fixed points in S^{n-1}_∞.

Remarks. 1. Example 1 is elliptic. Example 2 is parabolic or elliptic if $\lambda = 1$, and is loxodromic if $\lambda \neq 1$.

2. If h and k are conjugate in $\mathrm{Isom}(\mathbb{H}^n)$, then h is elliptic (respectively, parabolic, loxodromic) if and only if k is elliptic (respectively, parabolic, loxodromic).

3. Some authors use the term 'hyperbolic' instead of loxodromic. This is confusing, since one can then talk about non-hyperbolic isometries of hyperbolic space.

Theorem 4.1. *Every isometry $h\colon \mathbb{H}^n \to \mathbb{H}^n$ is either elliptic, parabolic or loxodromic.*

We need to show two things: that every hyperbolic isometry has a fixed point somewhere either in hyperbolic space or in the sphere at infinity and that if it has no fixed in hyperbolic space, then it has at most two fixed points on the sphere at infinity.

Proposition 4.2. *Any isometry $U^n \to U^n$ fixing ∞ is of the form $x \mapsto \lambda A x + b$, where $\lambda \in \mathbb{R}_{>0}$, $b \in \mathbb{R}^{n-1} \times \{0\}$ and A is an orthogonal map fixing e_n.*

Proof. Let $h: U^n \to U^n$ be an isometry fixing ∞. It sends 0 to some point b. Let g be the translation $x \mapsto x + b$. Then $g^{-1}h$ fixes 0 and ∞. It therefore preserves the unique geodesic α between them. It acts as an isometry on α (which is isometric to $\mathbb{R}$). It maps some point x on α to λx. Let f be the map $x \mapsto \lambda x$. Then $f^{-1}g^{-1}h$ fixes x and acts on T_xU^n via some orthogonal map A fixing the e_n direction. This orthogonal map A is an isometry of U^n. By Theorem 1.5, $f^{-1}g^{-1}h$ equals A. Therefore, h is the map $x \mapsto \lambda Ax + b$. □

Corollary 4.3. *A non-elliptic isometry h which fixes at least two points on S^{n-1}_∞ is conjugate to an isometry as in Example 2, with $\lambda \neq 1$. In particular, h is loxodromic.*

Proof. By conjugating the isometry, we may assume that the fixed points are at 0 and ∞. By Proposition 4.2, this isometry is of the form $x \mapsto \lambda Ax + b$, where b must be zero. If $\lambda = 1$, then it fixes all points on the e_n axis and hence h is elliptic. Therefore, $\lambda \neq 1$ and so the isometry is loxodromic. □

Proof of Theorem 4.1. Let h be a non-elliptic isometry. Corollary 4.3 implies it has most two fixed points on S^{n-1}_∞. We must show that it has at least one fixed point on S^{n-1}_∞. This is a special case of the Brouwer fixed point theorem which asserts that any continuous map from the closed unit ball in $\mathbb{R}^n$ to itself has a fixed point. Instead of quoting this, we'll prove it in this case.

Consider the displacement function

$$\begin{aligned} \mathbb{H}^n &\xrightarrow{f} \mathbb{R}_{\geq 0} \\ x &\mapsto d_{hyp}(h(x), x). \end{aligned}$$

This is a continuous function. Either

1. the infimum of f is attained and is zero, or
2. the infimum of f is not attained, or
3. the infimum of f is attained and is non-zero.

Case 1. The point $x \in \mathbb{H}^n$ where the infimum is attained is a fixed point for h, and therefore h is elliptic.

Case 2. Then there is a sequence of points $x_1, x_2, \ldots$ in $\mathbb{H}^n$ such that $f(x_i) \to \inf(f)$. This sequence has a convergent subsequence in $\mathbb{H}^n \cup S^{n-1}_\infty$, since this is compact. Pass to this subsequence. The limit point x cannot lie in $\mathbb{H}^n$, since then $f(x) = \inf(f)$ and the infimum is attained. Therefore, $x \in S_\infty^{n-1}$. We will show that x is a fixed point for h.

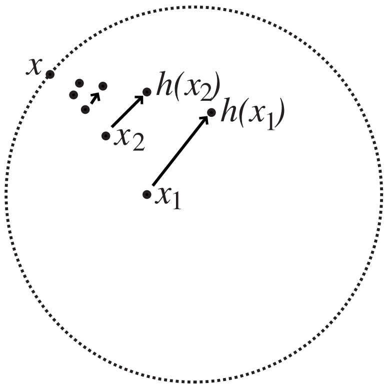


Figure 7.

Let I be the interval $[0, d_{hyp}(x_i, h(x_i))]$, and let $\alpha_i\colon I \to \mathbb{H}^n$ be the unit speed geodesic between x_i and $h(x_i)$. The sequence $d_{hyp}(x_i, h(x_i))$ is bounded above by some number M, say. Let 0 be the origin in D^n.

Since the points x_i are tending to a point on S_∞^{n-1}, the distance $d_{hyp}(x_i, 0) \to \infty$, by Corollary 3.7. By the triangle inequality,

$$d_{hyp}(\alpha_i(t), 0) \geq d_{hyp}(x_i, 0) - d_{hyp}(x_i, \alpha(t)) \geq d_{hyp}(x_i, 0) - M.$$

Therefore,

$$\inf_{t \in I} d_{hyp}(\alpha_i(t), 0) \to \infty$$

as $i \to \infty$. So,

$$\inf_{t \in I} d_{Eucl}(\alpha_i(t), 0) \to 1$$

as $i \to \infty$. Now,

$$1 = ||\alpha_i'(t)||_{hyp} = ||\alpha_i'(t)||_{Eucl} \left(\frac{2}{1 - [d_{Eucl}(\alpha_i(t), 0)]^2} \right).$$

So,

$$\sup_{t \in I} ||\alpha_i'(t)||_{Eucl} \to 0$$

as $i \to \infty$. Thus,

$$d_{Eucl}(x_i, h(x_i)) \leq \int_I ||\alpha_i'(t)||_{Eucl}\, dt \to 0$$

as $i \to \infty$. So, x is a fixed point of h.

Case 3. Let $x \in \mathbb{H}^n$ be a point where $\inf(f)$ is attained. We claim that the geodesic α through x and $h(x)$ is invariant under h. The endpoints of α on S^{n-1}_∞ are therefore preserved. They are not permuted, since otherwise h would have a fixed point in α. Hence, h fixes these points on S^{n-1}_∞. This will prove the theorem.

Suppose that α is not invariant under h. The geodesics α and $h(\alpha)$ meet at $h(x)$. The angle between them is neither 0 or π (since that would imply that α was preserved by h).

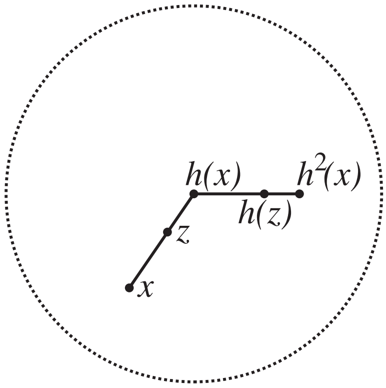


Figure 8.

Let z be any point on α between x and $h(x)$. Then

$$f(z) = d_{hyp}(z, h(z)) < d_{hyp}(z, h(x)) + d_{hyp}(h(x), h(z))$$

$$= d_{hyp}(z, h(x)) + d_{hyp}(x, z) = d_{hyp}(x, h(x)) = f(x).$$

Note that the inequality is strict because the angle between α and $h(\alpha)$ is not π. This contradicts the assumption that $\inf(f)$ is attained at x. □

We now know that every hyperbolic isometry is elliptic, parabolic or loxodromic. We also know from Corollary 4.3 that any loxodromic isometry is conjugate to one as in Example 2. What about elliptic and parabolic isometries?

Proposition 4.4. *An elliptic isometry is conjugate in* $\mathrm{Isom}(\mathbb{H}^n)$ *to an isometry as in Example 1.*

Proof. Let h be an elliptic isometry and let x be a fixed point for h in D^n. Pick an isometry k which takes x to 0. Then khk^{-1} fixes 0. It therefore acts on T_0D^n

via an orthogonal map A. Let $A\colon D^n \to D^n$ be the isometry as in Example 1. By Theorem 1.5 of the Introduction to Riemannian manifolds, $khk^{-1} = A$. □

Proposition 4.5. *A parabolic isometry h is conjugate in* $\mathrm{Isom}(\mathbb{H}^n)$ *to an isometry as in Example 2, with $\lambda = 1$.*

Proof. By conjugating the isometry, we may assume that it fixes ∞ in the upper-half space model. By Proposition 4.2, it therefore acts as $x \mapsto \lambda Ax + b$ as in Example 2. We will show that $\lambda = 1$. Now, khk^{-1} is parabolic and so has no fixed point in ∂U^n. So,

$$x = \lambda Ax + b$$

has no solution. Therefore, $\det(\lambda A - I) = 0$, which means that λ^{-1} is a root of the characteristic polynomial for A. Hence, λ^{-1} is an eigenvalue of the orthogonal map A. So, $\lambda = 1$, since it is positive. □

Each isometry $\mathbb{H}^n \to \mathbb{H}^n$ extends to a homeomorphism $\mathbb{H}^n \cup S^{n-1}_\infty \to \mathbb{H}^n \cup S^{n-1}_\infty$. Therefore, this defines an *extension homomorphism*

$$\mathrm{Isom}(\mathbb{H}^n) \to \mathrm{Homeo}(S^{n-1}_\infty),$$

where $\mathrm{Homeo}(S^{n-1}_\infty)$ is the group of homeomorphisms of S^{n-1}_∞.

Proposition 4.6. *This homomorphism is injective.*

Proof. Suppose that an isometry h fixes S^{n-1}_∞. If h is elliptic, then, by Proposition 4.4, it is conjugate to an isometry as in Example 1. This must be the identity on $\mathbb{H}^n$, and therefore, h is the identity on $\mathbb{H}^n$.

Suppose now that h is non-elliptic. By Corollary 4.3, h fixes exactly two points on S^{n-1}_∞, which is a contradiction. □

What this means is that a hyperbolic isometry is determined by its action on the sphere at infinity.

5. $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$

Definition. $\mathrm{SL}(2,\mathbb{C})$ is the group of 2×2 matrices with entries in $\mathbb{C}$ and with determinant one. The group $\mathrm{PSL}(2,\mathbb{C})$ is the quotient of $\mathrm{SL}(2,\mathbb{C})$ by the normal subgroup $\{\mathrm{id}, -\mathrm{id}\}$. The groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ are defined similarly.

There is a well-known relationship between $\mathrm{PSL}(2,\mathbb{C})$ and Möbius maps. Associated with each element

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}),$$

there is a Möbius map

$$\begin{aligned} \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} \\ z &\mapsto \frac{az+b}{cz+d}. \end{aligned}$$

This establishes an isomorphism between $\mathrm{PSL}(2,\mathbb{C})$ and the group of Möbius maps (where the group operation in the latter is composition of maps).

Theorem 5.1. $\mathrm{Isom}^+(\mathbb{H}^3)$ *is isomorphic to* $\mathrm{PSL}(2,\mathbb{C})$.

Identify ∂U^3 with $\mathbb{C}$, and S^2_∞ with $\mathbb{C} \cup \{\infty\}$. Consider the homomorphism

$$\mathrm{Isom}^+(\mathbb{H}^3) \to \mathrm{Homeo}(S^2_\infty) \cong \mathrm{Homeo}(\mathbb{C} \cup \infty),$$

which is injective by Proposition 4.6. Also, the map

$$\mathrm{PSL}(2,\mathbb{C}) \to \mathrm{Homeo}(\mathbb{C} \cup \{\infty\})$$

sending each matrix to its Möbius map is injective. Our aim is to show that the images of these two homomorphisms coincide.

Lemma 5.2. *The map* $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ *sending* $z \mapsto 1/z$ *is the extension of an orientation-preserving hyperbolic isometry.*

Proof. The isometry of D^3 will be the rotation ρ of angle π around the x-axis. This has the following effect on $\mathbb{C} \cup \{\infty\}$

$$\mathbb{C} \cup \{\infty\} \xrightarrow{I^{-1}} S^2_\infty \xrightarrow{\rho} S^2_\infty \xrightarrow{I} \mathbb{C} \cup \{\infty\}$$

sending a complex number z to a complex number z'. We must check that $z' = 1/z$. Clearly, $\arg(z') = -\arg(z)$.

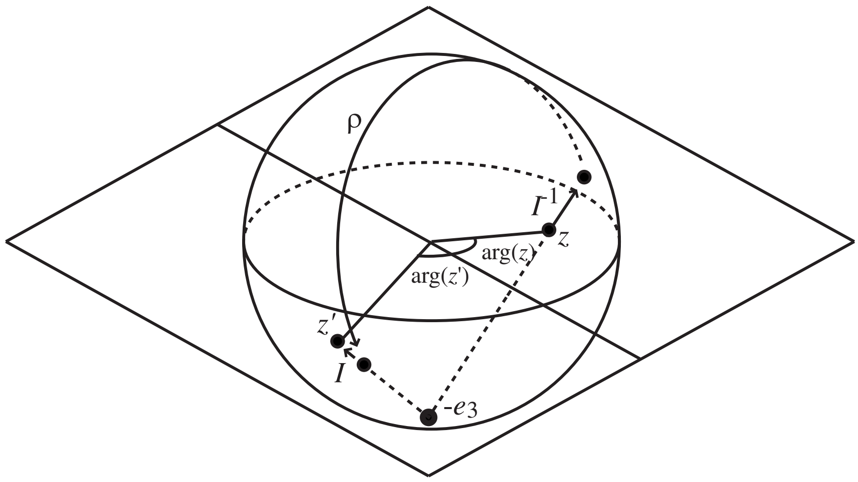


Figure 9.

We just have to check that $|z'| = 1/|z|$. Recall the definition of the map

$$\mathbb{R}^3 - \{-e_3\} \xrightarrow{I} \mathbb{R}^3 - \{-e_3\}$$
$$x \mapsto 2\frac{x+e_3}{[d_{Eucl}(x+e_3,0)]^2} - e_3.$$

Suppose that $x = (x_1, x_2, x_3) \in S^2_\infty$. So $x_1^2 + x_2^2 + x_3^2 = 1$. Also, $I(x) = (z_1, z_2, 0) = z \in \mathbb{C}$. Therefore

$$|z|^2 + 1 = [d_{Eucl}(z, -e_3))]^2 = 4[d_{Eucl}(x, -e_3)]^{-2}$$

$$= 4/[x_1^2 + x_2^2 + (x_3+1)^2] = 4/[2x_3+2] = 2/[x_3+1],$$

and so

$$|z|^2 = \frac{1-x_3}{1+x_3}.$$

Therefore, changing x_3 to $-x_3$ (which is the effect of ρ) changes $|z|^2$ to $|z|^{-2}$. Hence, $|z'| = |z|^{-1}$. $\square$

Lemma 5.3. *Given any three distinct points x_1, x_2 and x_3 in $\mathbb{C} \cup \{\infty\}$, there is a Möbius map h such that $h(x_1) = 0$, $h(x_2) = 1$, $h(x_3) = \infty$.*

Proof. Consider the case where x_1, x_2 and x_3 are all in $\mathbb{C}$. Use the map

$$z \mapsto \left(\frac{x_2 - x_3}{x_2 - x_1}\right)\left(\frac{z - x_1}{z - x_3}\right).$$ $\square$

Lemma 5.4. *Every element of* $\mathrm{Isom}^+(\mathbb{H}^3)$ *fixes some point on* S^2_∞.

Proof. This is true by definition if h is parabolic or loxodromic. If h is elliptic, then by Proposition 4.4, it is conjugate to an isometry as in Example 1. But any element of $SO(3)$ is a rotation which has at least two fixed points on S^2_∞. □

Proof of Theorem 5.1. We first show that every Möbius map is the extension of an orientation-preserving hyperbolic isometry. Any Möbius map can be expressed as a composition of the following maps

$$z \mapsto a_1 z$$

$$z \mapsto z + a_2$$

$$z \mapsto 1/z$$

The first and second of these are extensions of Example 2. Both of these are orientation-preserving. The third is an extension of an orientation-preserving elliptic isometry by Lemma 5.2.

We now show that every orientation-preserving hyperbolic isometry h extends to a Möbius map. By Lemma 5.4, h has a fixed point on S^2_∞. By Lemma 5.3, there is a Möbius map k sending this fixed point to ∞. So, khk^{-1} is an orientation-preserving isometry fixing ∞. So, it is of the form $z \mapsto \lambda Az + b$, as in Proposition 4.2. Since khk^{-1} is orientation-preserving, A is a rotation about e_n. So, khk^{-1} acts as $z \mapsto az + b$ ($a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$) which is a Möbius map. □

Theorem 5.5. $\mathrm{Isom}^+(\mathbb{H}^2)$ *is isomorphic to* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. Note that $\mathrm{PSL}(2, \mathbb{R})$ is the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ which leaves $\mathbb{R} \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\}$ invariant and preserves its orientation. Therefore, $\mathrm{PSL}(2, \mathbb{R})$ contains $\mathrm{Isom}^+(\mathbb{H}^2)$. To establish the opposite inclusion, we check that any orientation-preserving isometry of $\mathbb{H}^2$ extends to an orientation-preserving of $\mathbb{H}^3$. However, the orientation-preserving isometries of $\mathbb{H}^2$ are generated by the orientation-preserving isometries in Examples 1 and 2, and these extend to $\mathbb{H}^3$. □