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$\mathcal{PL}_0$ as a convex polyhedron in $\mathbb{R}^n$, so that changes of stereographic coordinates are piecewise projective, although this finite-dimensional picture cannot be strictly correct, since there is no fixed subdivision sufficient to make all coordinate changes.

**Corollary 9.7.3.** $\mathcal{PL}_0(S)$ is homeomorphic to a sphere.

**Proof that 9.7.2 implies 9.7.3.** Let $\gamma \in \mathcal{PL}_0(S)$ be any essentially complete lamination. Let $\tau$ be any train track carrying $\gamma$. Then $\mathcal{PL}_0(S)$ is the union of two coordinate systems $V_\tau \cup S_\tau$, which are mapped to convex sets in Euclidean space. If $\Delta_\gamma \neq \gamma$, nonetheless the complement of $\Delta_\gamma$ in $V_\tau$ is homeomorphic to $V_\tau - \gamma$, so $\mathcal{PL}_0(S)$ is homeomorphic to the one-point compactification of $S_\gamma$.

**Corollary 9.7.4.** When $\mathcal{PL}_0(S)$ has dimension greater than 1, it does not have a projective structure. (In other words, the pieces in changes of coordinates have not been eliminated.)

**Proof that 9.7.3 implies 9.7.4.** The only projective structure on $S^n$, when $n > 1$, is the standard one, since $S^n$ is simply connected. The binary relation of antipodality is natural in this structure. What would be the antipodal lamination for a simple closed curve $\alpha$? It is easy to construct a diffeomorphism fixing $\alpha$ but moving any other given lamination. (If $i(\gamma, \alpha) \neq 0$, the Dehn twist around $\alpha$ will do.)

**Remark.** When $\mathcal{PL}_0(S)$ is one-dimensional (that is, when $S$ is the punctured torus or the quadruply punctured sphere), the PIP structure does come from a projective structure, equivalent to $\mathbb{R}P^1$. The natural transformations of $\mathcal{PL}_0(S)$ are necessarily integral—in $\text{PSL}_2(\mathbb{Z})$. 
Proof of 9.7.2. Don’t blink. Let $\gamma$ be essentially complete. For each region $R_i$ of $S - \gamma$, consider a smaller region $r_i$ of the same shape but with finite points, rotated so its points alternate with cusps of $R_i$ and pierce very slightly through the sides of $R_i$, ending on a leaf of $\gamma$.

By 9.5.4, 9.5.2 and 9.3.9, both ends of each leaf of $\gamma$ are dense in $\gamma$, so the regions $r_i$ separate leaves of $\gamma$ into arcs. Each region of $S - \gamma - U_i r_i$ must be a rectangle with two edges on $\partial r_i$ and two on $\gamma$, since $r_i$ covers the “interesting” part of $R_i$. (Or, prove this by area, $\chi$). Collapse all rectangles, identifying the $r_i$ edges with each other, and obtain a surface $S'$ homotopy-equivalent to $S$, made of $U_i r_i$, where $\partial r_i$ projects to a train track $\tau$. (Equivalently, one may think of $S - U_i r_i$ as made of very wide corridors, with the horizontal direction given approximately by $\gamma$).
If we take shrinking sequences of regions $r_{i,j}$ in this manner, we obtain a sequence of train tracks $\tau_j$ which obviously have the property that $\tau_j$ carries $\tau_k$ when $j > k$.

Let $\gamma' \in \mathcal{PL}_0(S) - \Delta_{\gamma}$ be any lamination not topologically equivalent to $\gamma$. From the density in $\gamma$ of ends of leaves of $\gamma$, it follows that whenever leaves of $\gamma$ and $\gamma'$ cross, they cross at an angle. There is a lower bound to this angle. It also follows that $\gamma \cup \gamma'$ cuts $S$ into pieces which are compact except for cusps of $S$.

When $R_i$ is an asymptotic triangle, for instance, it contains exactly one region of $S - \gamma - \gamma'$ which is a hexagon, and all other regions of $S - \gamma - \gamma'$ are rectangles. For sufficiently high $j$, the $r_{ij}$ can be isotoped, without changing the leaves of $\gamma$ which they touch, into the complement of $\gamma'$. It follows that $\gamma'$ projects nicely to $\tau_j$. 
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Stereographic coordinates give a method of computing and understanding intersection number. The transverse measure for $\gamma$ projects to a “tangential” measure $\nu_\gamma$ on each of the train tracks $\tau_i$: i.e., $\nu_\gamma(b)$ is the $\gamma$-transverse length of the sides of the rectangle projecting to $b$.

It is clear that for any $\alpha \in \mathcal{ML}_0$ which is determined by a measure $\mu_\alpha$ on $\tau_i$

\[ i(\alpha, \gamma) = \sum_b \mu_\alpha(b) \cdot \nu_\gamma(b). \]

Thus, in the coordinate system $V_{\tau_i}$ in $\mathcal{ML}_0$, intersection with $\gamma$ is a linear function.

To make this observation more useful, we can reverse the process of finding a family of “transverse” train tracks $\tau_i$ depending on a lamination $\gamma$. Suppose we are given an essentially complete train track $\tau$, and a non-negative function (or “tangential” measure) $\nu$ on the branches of $b$, subject only to the triangle inequalities

\[ a + b - c \geq 0 \quad a + c - b \geq 0 \quad b + c - a \geq 0 \]

whenever $a, b$ and $c$ are the total $\nu$-lengths of the sides of any triangle in $S - \tau$. We shall construct a “train track” $\tau^*$ dual to $\tau$, where we permit regions of $S - \tau^*$ to be bigons as well as ordinary types of admissible regions—let us call $\tau^*$ a bigon track.
\( \tau^* \) is constructed by shrinking each region \( R_i \) of \( S - \tau \) and rotating to obtain a region \( R'_i \subset R_i \) whose points alternate with points of \( R_i \). These points are joined using one more branch \( b^* \) crossing each branch \( b \) of \( \tau \); branches \( b_1^* \) and \( b_2^* \) are confluent at a vertex of \( R^* \) whenever \( b_1 \) and \( b_2 \) lie on the same side of \( R \). Note that there is a bigon in \( S - \tau^* \) for each switch in \( \tau \).

The tangential measure \( \nu \) for \( \tau \) determines a transverse measure defined on the branches of \( \tau^* \) of the form \( b^* \). This extends uniquely to a transverse for \( \tau^* \) when \( S \) is not a punctured torus.

When \( S \) is the punctured torus, then \( \tau \) must look like this, up to the homeomorphism (drawn on the abelian cover of \( T - p \)): 

\[
\begin{align*}
A &= \frac{1}{2}(b+c-a) \\
b &= \frac{1}{2}(a+c-b) \\
c &= \frac{1}{2}(a+b-c)
\end{align*}
\]
Note that each side of the punctured bigon is incident to each branch of $\tau$. Therefore, the tangential measure $\nu$ has an extension to a transverse measure $\nu^*$ for $\tau^*$, which is unique if we impose the condition that the two sides of $R^*$ have equal transverse measure.

A transverse measure on a bigon track determines a measured geodesic lamination, by the reasoning of 8.9.4. When $\tau$ is an essentially complete train track, an open subset of $\mathcal{ML}_0$ is determined by a function $\mu$ on the branches of $\tau$ subject to a condition for each switch that

$$\sum_{b \in I} \mu(b) = \sum_{b \in \partial} \mu(b),$$

where $I$ and $\partial$ are the sets of “incoming” and “outgoing” branches. Dually, “tangential” measure $\nu$ on the branches of $\tau$ determines an element of $\mathcal{ML}_0$ (via $\nu^*$), but two functions $\nu$ and $\nu'$ determine the same element if $\nu$ is obtained from $\nu'$ by a process of adding a constant to the incoming branches of a switch, and subtracting the same constant from the outgoing branches—or, in other words, if $\nu - \nu'$ annihilates all transverse measures for $\tau$ (using the obvious inner product $\nu \cdot \mu = \sum \nu(b)\mu(b)$). In fact, this operation on $\nu$ merely has the effect of switching “trains” from one side of a bigon to the other.
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(Some care must be taken to obtain $\nu'$ from $\nu$ by a sequence of elementary "switching" operations without going through negative numbers. We leave this as an exercise to the reader.)

Given an essentially complete train track $\tau$, we now have two canonical coordinate systems $V_\tau$ and $V^*_\tau$ in $\mathcal{ML}_0$ or $\mathcal{PL}_0$. If $\gamma \in V_\tau$ and $\gamma^* \in V^*_\tau$ are defined by measures $\mu_\gamma$ and $\nu_{\gamma^*}$ on $\tau$, then $i(\gamma, \gamma^*)$ is given by the inner product

$$i(\gamma, \gamma^*) = \sum_{b \in \tau} \mu_\gamma(b)\nu_{\gamma^*}(b).$$

To see this, consider the universal cover of $S$. By an Euler characteristic or area argument, no path on $\tilde{\tau}$ can intersect a path on $\tilde{\tau}^*$ more than once. This implies the formula when $\gamma$ and $\gamma'$ are simple geodesics, hence, by continuity, for all measured geodesic laminations.

**Proposition 9.7.4.** Formula 9.7.3 holds for all $\gamma \in V_\tau$ and $\gamma^* \in V^*_\tau$. Intersection number is a bilinear function on $V_\tau \times V^*_\tau$ (in $\mathcal{ML}_0$).

This can be interpreted as a more intrinsic justification for the linear structure on the coordinate systems $V_\tau$—the linear structure can be reconstructed from the embedding of $V_\tau$ in the dual space of the vector space with basis $\gamma^* \in V^*_\tau$.

**Corollary 9.7.5.** If $\gamma, \gamma' \in \mathcal{ML}_0$ are not topologically conjugate and if at least one of them is essentially complete, then there are neighborhoods $U$ and $U'$ of $\gamma$ and $\gamma'$ with linear structures in which intersection number is bilinear.

**Proof.** Apply 9.7.4 to one of the train tracks $\tau_i$ constructed in 9.7.2.

**Remark.** More generally, the only requirement for obtaining this local bilinearity near $\gamma$ and $\gamma'$ is that the complementary regions of $\gamma \cup \gamma'$ are "atomic" and that $S - \gamma$ have no closed non-peripheral curves. To find an appropriate $\tau$, simply burrow out regions of $r_i$, "transverse" to $\gamma$ with points going between strands of $\gamma'$, so the regions $r_i$ cut all leaves of $\gamma$ into arcs. Then collapse to a train track carrying $\gamma'$ and "transverse" to $\gamma$, as in 9.7.2.
What is the image of $\mathbb{R}^n$ of stereographic coordinates $S_\gamma$ for $\mathcal{ML}_0(S)$? To understand this, consider a system of train tracks

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_k \rightarrow \cdots$$

defining $S_\gamma$. A “transverse” measure for $\tau_i$ pushes forward to a “transverse” measure for $\tau_j$, for $j > i$. If we drop the restriction that the measure on $\tau_i$ is non-negative, still it often pushes forward to a positive measure on $\tau_j$. The image of $S_\gamma$ is the set of such arbitrary “transverse” measures on $\tau_1$ which eventually become positive when pushed far enough forward.

For $\gamma' \in \Delta_\gamma$, let $\nu_{\gamma'}$ be a “tangential” measure on $\tau_1$ defining $\gamma'$.

**Proposition 9.7.6.** The image of $S_\gamma$ is the set of all “transverse,” not necessarily positive, measures $\mu$ on $\tau_1$ such that for all $\gamma' \in \Delta_\gamma$, $\nu_{\gamma'} \cdot \mu > 0$.

(Note that the functions $\nu_{\gamma'} \cdot \mu$ and $\nu_{\gamma''} \cdot \mu$ are distinct for $\gamma' \neq \gamma''$.)

In particular, note that if $\Delta_\gamma = \gamma$, the image of stereographic coordinates for $\mathcal{ML}_0$ is a half-space, or for $\mathcal{PL}_0$ the image is $\mathbb{R}^n$. If $\Delta_\gamma$ is a $k$-simplex, then the image of $S_\gamma$ for $\mathcal{PL}_0$ is of the form $\text{int}(\Delta^k) \times \mathbb{R}^{n-k}$. (This image is defined only up to projective equivalence, until a normalization is made.)
PROOF. The condition that $\nu_{\gamma'} \cdot \mu > 0$ is clearly necessary: intersection number $i(\gamma', \gamma'')$ for $\gamma' \in \Delta_\gamma$, $\gamma'' \in S_\gamma$ is bilinear and given by the formula

$$i(\gamma', \gamma'') = \nu_{\gamma'} \cdot \mu_{\gamma''}.$$  

Consider any transverse measure $\mu$ on $\tau_1$ such that $\mu$ is always non-positive when pushed forward to $\tau_i$. Let $b_i$ be a branch of $\tau_i$ such that the push-forward of $\mu$ is non-positive on $b_i$. This branch $b_i$, for high $i$, comes from a very long and thin rectangle $\rho_i$. There is a standard construction for a transverse measure coming from a limit of the average transverse counting measures of one of the sides of $\rho_i$. To make this more concrete, one can map $\rho_i$ in a natural way to $\tau_j$ for $j \leq i$.

(In general, whenever an essentially complete train track $\tau$ carries a train track $\sigma$, then $\sigma^*$ carries $\tau^*$

$$\sigma \to \tau$$

$$\sigma^* \leftarrow \tau^*.$$  

To see this, embed $\sigma$ in a narrow corridor around $\tau$, so that branches of $\tau^*$ do not pass through switches of $\sigma$. Now $\sigma^*$ is obtained by squeezing all intersections of branches of $\tau^*$ with a single branch of $\sigma$ to a single point, and then eliminating any bigons contained in a single region of $S - \sigma$.)

On $\tau_i^*$, $\rho_i$ is a finite but very long path. The average number of times $\rho_i$ tranverses a branch of $\tau_i^*$ gives a function $\nu_i$ which almost satisfies the switch condition, but not quite. Passing to a limit point of $\{\nu_i\}$ one obtains a “transverse” measure $\nu$ for $\tau_i^*$, whose lamination topologically equals $\gamma$, since it comes from a transverse measure on $\tau_i^*$, for all $i$. Clearly $\nu \cdot \mu \leq 0$, since $\nu_i$ comes from a function supported on a single branch $b_i^*$ of $\tau_i^*$, and $\mu(b_i) < 0$.  

For $\gamma \in \mathcal{ML}_0$ let $\mathcal{Z}_\gamma \subset \mathcal{ML}_0$ consist of $\gamma'$ such that $i(\gamma, \gamma') = 0$. Let $C_\gamma$ consist of laminations $\gamma'$ not intersecting $\gamma$, i.e., such that support of $\gamma'$ is disjoint from the support of $\gamma$. An arbitrary element of $\mathcal{Z}_\gamma$ is an element of $C_\gamma$, together with some
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measure on $\gamma$. The same symbols will be used to denote the images of these sets in $\mathcal{PL}_0(S)$.

**Proposition 9.7.6.** The intersection of $Z_\gamma$ with any of the canonical coordinate systems $X$ containing $\gamma$ is convex. (In $\mathcal{ML}_0$ or $\mathcal{PL}_0$.)

**Proof.** It suffices to give the proof in $\mathcal{ML}_0$. First consider the case that $\gamma$ is a simple closed curve and $X = V_\tau$, for some train track $\tau$ carrying $\gamma$. Pass to the cylindrical covering space $C$ of $S$ with fundamental group generated by $\gamma$. The path of $\gamma$ on $C$ is embedded in the train track $\tilde{\tau}$ covering $\tau$. From a “transverse” measure $m$ on $\tilde{\tau}$, construct corridors on $C$ with a metric giving them the proper widths.

For any subinterval $I$ of $\gamma$, let $\text{nxr}(I)$ and $\text{nxr}(I)$ be (respectively) the net right hand exiting and the net left hand exiting in the corresponding to $I$; in computing this, we weight entrances negatively. (We have chosen some orientation for $\gamma$). Let $i(I)$ be the initial width of $I$, and $f(I)$ be the final width.

If the measure $m$ comes from an element $\gamma'$, then $\gamma' \in Z_\gamma$ if and only if there is no “traffic” entering the corridor of $\gamma$ on one side and exiting on the other. This implies the inequalities

$$i(I) \geq \text{nxr}(I)$$

and

$$i(I) \geq \text{nxr}(I)$$

for all subintervals $I$.  

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It also implies the equation

\[ \text{nxl}(\gamma) = 0, \]

so that any traffic travelling once around the corridor returns to its initial position. (Otherwise, this traffic would spiral around to the left or right, and be inexorably forced off on the side opposite to its entrance.)

Conversely, if these inequalities hold, then there is some trajectory going clear around the corridor and closing up. To see this, begin with any cross-section of the corridor. Let \( x \) be the supremum of points whose trajectories exit on the right. Follow the trajectory of \( x \) as far as possible around the corridor, always staying in the corridor whenever there is a choice.

The trajectory can never exit on the left—otherwise some trajectory slightly lower would be forced to enter on the right and exit on the left, or vice versa. Similarly, it can’t exit on the right. Therefore it continues around until it closes up.
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Thus when $\gamma$ is a simple closed curve, $Z_\gamma \cap V_+$ is defined by linear inequalities, so it is convex.

Consider now the case $X = V_\tau$ and $\gamma$ is connected but not a simple geodesic. Then $\gamma$ is associated with some subsurface $M_\gamma \subset S$ with geodesic boundary defined to be the minimal convex surface containing $\gamma$. The set $C_\gamma$ is the set of laminations not intersecting $\text{int}(M_\gamma)$. It is convex in $V_\tau$, since

$$C_\gamma = \bigcap \{Z_\alpha | \alpha \text{ is a simple closed curve} \subset \text{int}(M_\gamma)\}.$$ 

A general element $\gamma'$ of $Z_\gamma$ is a measure on $\gamma \cup \gamma''$, so $Z_\gamma$ consists of convex combinations of $\Delta_\gamma$ and $C_\gamma$: hence, it is convex.

If $\gamma$ is not connected, then $Z_\gamma$ is convex since it is the intersection of $\{Z_{\gamma_i}\}$, where the $\gamma_i$ are the components of $\gamma$.

The case $X$ is x stereographic coordinate system follows immediately. When $X = V_\tau^*$, consider any essentially complete $\gamma \in V_\tau$. From 9.7.5 it follows that $V_\tau^*$ is linearly embedded in $S_\gamma$. (Or more directly, construct a train track (without bigons) carrying $\tau^*$; or, apply the preceding proof to bigon track $\tau^*$.)

**Remark.** Note that when $\gamma$ is a union of simple closed curves, $C_\gamma$ in $\mathcal{PL}_0(S)$ is homeomorphic to $\mathcal{PL}_0(S - \gamma)$, regarded as a complete surface with finite area—i.e., $C_\gamma$ is a sphere. When $\gamma$ has no component which is a simple closed curve, $C_\gamma$ is convex. Topologically, it is the join of $\mathcal{PL}_0(S - \bigcup S_{\gamma_i})$ with the simplex of measures on the boundary components of the $S_{\gamma_i}$, where the $S_{\gamma_i}$ are subsurfaces associated with the components $\gamma_i$ of $\gamma$.

Now we are in a position to form an image of the set of unrealizable laminations for $\rho \pi_1S$. Let $U_+ \subset \mathcal{PL}_0$ be the union of laminations containing a component of $\chi_+$ and define $U_-$ similarly, so that $\gamma$ is unrealizable if and only if $\gamma \in U_+ \cup U_-$. $U_+$ is a union of finitely many convex pieces, and it is contained in a subcomplex of $\mathcal{PL}_0$ of...
codimension at least one. It may be disjoint from $U_-$, or it may intersect $U_-$ in an interesting way.

**Example.** Let $S$ be the twice punctured torus. From a random essentially complete train track,

we compute that $\mathcal{ML}_0$ has dimension 4, so $\mathcal{PL}_0$ is homeomorphic to $S^3$. For any simple closed curve $\alpha$ on $S$, $C_\alpha$ is $\mathcal{PL}_0(S - \alpha)$,

where $S - \alpha$ is either a punctured torus union a (trivial) thrice punctured sphere, or a 4-times punctured sphere. In either case, $C_\alpha$ is a circle, so $Z_\alpha$ is a disk.

Here are some sketches of what $U_+$ and $U_-$ can look like.
Here is another example, where $S$ is a surface of genus 2, and $U_+(S) \cup U_-(S)$ has the homotopy type of a circle (although its closure is contractible):

In fact, $U_+ \cup U_-$ is made up of convex sets $Z_\gamma - C_\gamma$, with relations of inclusion as diagrammed:
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The closures all contain the element $\alpha$; hence the closure of the union is starlike:

\[ \gamma = \]

9.9. Ergodicity of the geodesic flow

We will prove a theorem of Sullivan (1979):

**Theorem 9.9.1.** Let $M^n$ be a complete hyperbolic manifold (of not necessarily finite volume). Then these four conditions are equivalent:

(a) The series

\[ \sum_{\gamma \in \pi_1 M^n} \exp\left( - (n-1) d(x_0, \gamma x_0) \right) \]

diverges. (Here, $x_0 \in H^n$ is an arbitrary point, $\gamma x_0$ is the image of $x_0$ under a covering transformation, and $d(,)$ is hyperbolic distance).

(b) The geodesic flow is not dissipative. (A flow $\phi_t$ on a measure space $(X, \mu)$ is dissipative if there exists a measurable set $A \subset X$ and a $T > 0$ such that $\mu(A \cap \phi_t(A)) = 0$ for $t > T$, and $X = \cup_{t \in (A)}$.)

(c) The geodesic flow on $T_1(M)$ is recurrent. (A flow $\phi_t$ on a measure space $(X, \mu)$ is recurrent when for every measure set $A \subset X$ of positive measure and every $T > 0$ there is a $t \geq T$ such that $\mu(A \cap \phi_t(A)) > 0$.)

(d) The geodesic flow on $T_1(M)$ is ergodic.

Note that in the case $M$ has finite volume, recurrence of the geodesic flow is immediate (from the Poincaré recurrence lemma). The ergodicity of the geodesic flow in this case was proved by Eberhard Hopf, in ???. The idea of (c) $\rightarrow$ (d) goes back to Hopf, and has been developed more generally in the theory of Anosov flows ??.
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**Corollary 9.9.2.** If the geodesic flow is not ergodic, there is a non-constant bounded superharmonic function on $M$.

**Proof of 9.9.2.** Consider the Green’s function $g(x) = \int_{d(x,x_0)}^{\infty} \sin h^{1-n} t \, dt$ for hyperbolic space. (This is a harmonic function which blows up at $x_0$.) By (a), the series $\sum_{\gamma \in \pi_1 M} g \circ \gamma$ converges to a function, invariant by $\gamma$, which projects to a Green’s function $G$ for $M$. The function $f = \arctan G$ (where $\arctan \infty = \pi/2$) is a bounded superharmonic function, since $\arctan$ is convex.

**Remark.** The convergence of the series (a) is actually equivalent to the existence of a Green’s function on $M$, and also equivalent to the existence of a bounded superharmonic function. See (Ahlfors, Sario) for the case $n = 2$, and [ ] for the general case.

**Corollary 9.9.3.** If $\Gamma$ is a geometrically tame Kleinian group, the geodesic flow on $T_1(H^n/\Gamma)$ is ergodic if and only if $L_{\Gamma} = S^2$.

**Proof of 9.9.3.** From 9.9.2 and 8.12.3.

**Proof of 9.9.1.** Sullivan’s proof of 9.9.1 makes use of the theory of Brownian motion on $M^n$. This approach is conceptually simple, but takes a certain amount of technical background (or faith). Our proof will be phrased directly in terms of geodesics, but a basic underlying idea is that a geodesic behaves like a random path: its future is “nearly” independent of its past.

![Diagram](9.9-2a)

(d) $\rightarrow$ (c). This is a general fact. If a flow $\phi_t$ is not recurrent, there is some set $A$ of positive measure such that only for $t$ in some bounded interval is $\mu(A \cap \phi_t(A)) > 0$. Then for any subset $B \subset A$ of small enough measure, $\cup_t \phi_t(B)$ is an invariant subset which is proper, since its intersection with $A$ is proper.
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(c) $\rightarrow$ (b). Immediate.

(b) $\rightarrow$ (a). Let $B$ be any ball in $H^n$, and consider its orbit $\Gamma B$ where $\Gamma = \pi_1 M$. For the series of (a) to diverge means precisely that the total apparent area of $\Gamma G$ as seen from a point $x_0 \in H^n$, (measured with multiplicity) is infinite.

In general, the underlying space of a flow is decomposed into two measurable parts, $X = D \cup R$, where $\phi_t$ is dissipative on $D$ (the union of all subsets of $X$ which eventually do not return) and recurrent on $R$. The reader may check this elementary fact. If the recurrent part of the geodesic flow is non-empty, there is some ball $B$ in $M^n$ such that a set of positive measure of tangent vectors to points of $B$ give rise to geodesics that intersect $B$ infinitely often. This clearly implies that the series of (a) diverges.

The idea of the reverse implication (a) $\rightarrow$ (b) is this: if the geodesic flow is dissipative there are points $x_0$ such that a positive proportion of the visual sphere is not covered infinitely often by images of some ball. Then for each “group” of geodesics that return to $B$, a definite proportion must eventually escape $\Gamma B$, because future and past are nearly independent. The series of (a) can be regrouped as a geometric progression, so it converges. We now make this more precise.

Recall that the term “visual sphere” at $x_0$ is a synonym to the “set of rays” emanating from $x_0$. It has a metric and a measure obtained from its identification with the unit sphere in the tangent space at $x_0$.

Let $x_0 \in M^n$ be any point and $B \subset M^n$ any ball. If a positive proportion of the rays emanating from $x_0$ pass infinitely often through $B$, then for a slightly larger ball $B'$, a definite proportion of the rays emanating from any point $x \in M^n$ spend an infinite amount of time in $B'$, since the rays through $x$ are parallel to rays through $x_0$. Consequently, a subset of $T_1(B')$ of positive measure consists of vectors whose geodesics spend an infinite total time in $T_1(B')$; by the Poincaré recurrence lemma, the set of such vectors is a recurrent set for the geodesic flow. (b) holds so (a) $\rightarrow$ (b) is valid in this case. To prove (a) $\rightarrow$ (b), it remains to consider the case that almost every ray from $x_0$ eventually escapes $B$; we will prove that (a) fails, i.e., the series of (a) converges.

Replace $B$ by a slightly smaller ball. Now almost every ray from almost every point $x \in M$ eventually escapes the ball. Equivalently, we have a ball $B \subset H^n$ such that for every point $x \in H^n$, almost no geodesic through $x$ intersects $\Gamma B$, or even $\Gamma(N_r(B))$, more than a finite number of times.

Let $x_0$ be the center of $B$ and let $\alpha$ be the infimum, for $y \in H^n$, of the diameter of the set of rays from $x_0$ which are parallel to rays from $y$ which intersect $B$. This infimum is positive, and very rapidly approached as $y$ moves away from $x_0$. 

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Let $R$ be large enough so that for every ball of diameter greater than $\alpha$ in the visual sphere at $x_0$, at most (say) half of the rays in this ball intersect $\Gamma N_\epsilon(B)$ at a distance greater than $R$ from $x_0$. $R$ should also be reasonably large in absolute terms and in comparison to the diameter of $B$.

Let $x_0$ be the center of $B$. Choose a subset $\Gamma' \subset \Gamma$ of elements such that: (i) for every $\gamma \in \Gamma$ there is a $\gamma' \in \Gamma'$ with $d(\gamma x_0, \gamma' x_0) < R$. (ii) For any $\gamma_1$ and $\gamma_2$ in $\Gamma'$, $d(\gamma_1 x_0, \gamma_2 x_0) \geq R$.

Any subset of $\Gamma$ maximal with respect to (ii) satisfies (i).

We will show that $\sum_{\gamma' \in \Gamma'} \exp(- (n-1) d(x_0, \gamma' x_0))$ converges. Since for any $\gamma'$ there are a bounded number of elements $\gamma \in \Gamma$ so that $d(\gamma x_0, \gamma' x_0) < R$, this will imply that the series of (a) converges.

Let $<$ be the partial ordering on the elements of $\Gamma'$ generated by the relation $\gamma_1 < \gamma_2$ when $\gamma_2 B$ eclipses $\gamma_1 B$ (partially or totally) as viewed from $x_0$; extend $<$ to be transitive.

Let us denote the image of $\gamma B$ in the visual sphere of $x_0$ by $B_\gamma$. Note that when $\gamma' < \gamma$, the ratio $\text{diam}(B_{\gamma'}) / \text{diam}(B_\gamma)$ is fairly small, less than $1/10$, say. Therefore $\cup_{\gamma' < \gamma} B_{\gamma'}$ is contained in a ball concentric with $B_\gamma$ of radius $10/9$ that of $B_\gamma$.

Choose a maximal independent subset $\Delta_1 \subset \Gamma'$ (this means there is no relation $\delta_1 < \delta_2$ for any $\delta_1, \delta_2 \in \Delta_1$). Do this by successively adjoining any $\gamma$ whose $B_\gamma$ has largest size among elements not less than any previously chosen member. Note that area $(\cup_{\delta \in \Delta} B_\delta) / \text{area}(\cup_{\gamma \in \Gamma'} B_\gamma)$ is greater than some definite (a priori) constant: $(9/10)^{n-1}$ in our example. Inductively define $\Gamma'_0 = \Gamma'$, $\gamma_{i+1}' = \Gamma'_i - \Delta_{i+1}$ and define $\Delta_{i+1} \subset \Gamma_i$ similarly to $\Delta_1$. Then $\Gamma' = \cup_{i=1}^\infty \Delta_i$.

For any $\gamma \in \Gamma'$, we can compare the set $B_\gamma$ of rays through $x_0$ which intersect $\gamma(B)$ to the set $C_\gamma$ of parallel rays through $\gamma X_0$.

Any ray of $B_\gamma$ which re-enters $\Gamma'(B)$ after passing through $\gamma'(B)$, is within $\epsilon$ of the parallel ray of $C_\gamma$ by that time. At most half of the rays of $C_\gamma$ ever enter $N_\epsilon(\Gamma' B)$.
The distortion between the visual measure of $B_\gamma$ and that of $C_\gamma$ is modest, so we can conclude that the set of reentering rays, $B_\gamma \cap \bigcup_{\gamma' < \gamma} B_{\gamma'}$, has measure less than $2/3$ the measure of $B_\gamma$.

We conclude that, for each $i$,

$$\begin{align*}
\text{area} \left( \bigcup_{\gamma \in \Gamma'_{i+1}} B_\gamma \right) - \text{area} \left( \bigcup_{\gamma \in \Gamma'_i} B_\gamma \right) & \geq 1/3 \text{ area} \left( \bigcup_{\delta \in \Delta_{i+1}} B_\delta \right) \\
& \geq 1/3 \cdot (9/10)^{n-1} \text{ area} \left( \bigcup_{\gamma \in \Gamma'_i} B_\gamma \right).
\end{align*}$$

The sequence $\{\text{area}(\bigcup_{\gamma \in \Gamma'_i} B_\gamma)\}$ decreases geometrically. This sequence dominates the terms of the series $\sum_i \text{area} \cup_{\delta \in \Delta_i} B_\delta = \sum_{\gamma \in \Gamma,} \text{area}(B_\gamma)$, so the latter converges, which completes the proof of (a) $\rightarrow$ (b).

(b) $\rightarrow$ (c). Suppose $R \subset T_1(M^n)$ is any recurrent set of positive measure for the geodesic flow $\phi_t$. Let $B$ be a ball such that $R \cap T_1(B)$ has positive measure. Almost every forward geodesic of a vector in $R$ spends an infinite amount of time in $B$. Let $A \subset T_1(B)$ consist of all vectors whose forward geodesics spend an infinite time in $B$ and let $\psi_t, t \geq 0$, be the measurable flow on $A$ induced from $\phi_t$ which takes a point leaving $A$ immediately back to its next return to $A$.

Since $\psi_t$ is measure preserving, almost every point of $A$ is in the image of $\psi_t$ for all $t$ and an inverse flow $\psi_{-t}$ is defined on almost all of $A$, so the definition of $A$ is unchanged under reversal of time. Every geodesic parallel in either direction to a geodesic in $A$ is also in $A$; it follows that $A = T_1(B)$. By the Poincaré recurrence lemma, $\psi_t$ is recurrent, hence $\phi_t$ is also recurrent.

(c) $\rightarrow$ (d). It is convenient to prove this in the equivalent form, that if the action of $\Gamma$ on $S^{n-1}_\infty \times S^{n-1}_\infty$ is recurrent, it is ergodic. “Recurrent” in this context means that for any set $A \subset S^{n-1}_\infty \times S^{n-1}_\infty$ of positive measure, there are an infinite number of elements $\gamma \in \Gamma$ such that $\mu(\gamma A \cap A) > 0$. Let $I \subset S^{n-1}_\infty \times S^{n-1}_\infty$ be any measurable set invariant by $\Gamma$. Let $-B_1$ and $B_2 \subset S^{n-1}$ be small balls. Let us consider what $I$ must look like near a general point $x = (x_1, x_2) \in B_1 \times B_2$. If $\gamma$ is a “large” element of $\Gamma$ such that $\gamma x$ is near $x$, then the preimage of $\gamma$ of a product of small $\epsilon$-ball around $\gamma x_1$ and $\gamma x_2$ is one of two types: it is a thin neighborhood of one of the factors, $(x_1 \times B_2)$ or $(B_1 \times x_2)$. ($\gamma$ must be a translation in one direction or the other along an axis from approximately $x_1$ to approximately $x_2$.) Since $\Gamma$ is recurrent, almost every point $x \in B_1 \times B_2$ is the preimage of elements $\gamma$ of both types, of an infinite number of
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points where \( I \) has density 0 or 1. Define

\[
f(x_1) = \int_{B_2} \chi_I(x_1, x_2) \, dx_2,
\]

where \( \chi_I \) is the characteristic function of \( I \), for \( x_1 \in B_1 \) (using a probability measure on \( B_2 \)). By the above, for almost every \( x_1 \) there are arbitrarily small intervals around \( x_1 \) such that the average of \( f \) in that interval is either 0 or 1. Therefore \( f \) is a characteristic function, so \( I \cap B_1 \times B_2 \) is of the form \( S \times B_2 \) (up to a set of measure zero) for some set \( S \subset B_1 \).

Similarly, \( I \) is of the form \( B_1 \times R \), so \( I \) is either \( \emptyset \times \emptyset \) or \( B_1 \times B_2 \) (up to a set of measure zero). \( \square \)