Representations into $SL_2(\mathbb{C})$; the players so far

Γ = finitely-generated group, generators $g_1, \dots , g_n$

$R(\Gamma) = \{\rho : \Gamma \to SL_2(\mathbb{C})$ homomorphisms $\}$ linear representations

$R(\Gamma) \hookrightarrow \mathbb{C}^{4n}$ (coords of generators - <u>not</u> a homom!)
is defined by (finitely many) polynomimal equations;
it is an **affine algebraic set**.

$V \subseteq \mathbb{C}^N$ alg set, the coordinate ring $\mathbb{C}[V] =$
$\{f : V \to \mathbb{C} \ : \ f = g|_V, g : \mathbb{C}^N \to \mathbb{C}$ polynomial fcn$\}$
If V is <u>irreducible</u> (no proper alg subset), then $\mathbb{C}[V]$ = integral domain.
$\mathbb{C}(V)$ = field of fractions = <u>function</u> <u>field</u> of V, think of as rational functions defined on open dense subsets $U \subseteq V$.

$R_0 \subseteq R(\Gamma)$, $\gamma \in \Gamma$, define $E_\gamma : R_0 \to SL_2(\mathbb{C})$ by $E_\gamma(\rho) = \rho(\gamma)$.
Entries of $\rho(\gamma)$ are polynomials in the entries of the $\rho(g_i)$, so are in $\mathbb{C}[R_0]$, so $E_\gamma \in SL_2(\mathbb{C}[R_0])$.

Characters:
$\chi_\rho : \Gamma \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma)) = \mathrm{tr}(E_\gamma(\rho) \equiv I_\gamma(\rho)$
Set of all characters = $X(\Gamma)$.
$I_\gamma : R(\Gamma) \to \mathbb{C}$, so $I_\gamma \in \mathbb{C}[R(\Gamma)]$

$T(\Gamma)$ = trace ring = subring of $\mathbb{C}[R(\Gamma)]$ generated by the $I\gamma$.
$T(\Gamma)$ is generated by I_γ,
$\gamma \in S = \{g_{i_1} \cdots g_{i_k} \ : \ i_1 < \cdots i_k\} = \{w_1, \dots , w_{2^n-1}\}$, so
$T(\Gamma) \hookrightarrow \mathbb{C}^{|S|} = \mathbb{C}^{2^n-1}$

$t : R(\Gamma) \to \mathbb{C}^{2^n-1}$, $t(\rho) = (I_{w_1}(\rho), \dots , I_{w_{2^n-1}}(\rho))$
$t(\rho) = t(\rho') \Leftrightarrow I_\gamma(\rho) = I_\gamma(\rho') \ \forall \ \gamma \Leftrightarrow \chi_\rho(\gamma) = \chi_{\rho'}(\gamma) \ \forall \ \gamma \Leftrightarrow \chi_\rho = \chi_{\rho'}$,
so $X(\Gamma) \leftrightarrow t(R(\Gamma)) \subseteq \mathbb{C}^{2^n-1}$.
$t(R(\Gamma))$ is an algebraic set; so $X(\Gamma)$ may be thought of as one, too.

The surjection $t : R(\Gamma) \to X(\Gamma)$ induces an injection
$J : \mathbb{C}[X(\Gamma)] \hookrightarrow \mathbb{C}[R(\Gamma)]$, with image generated by the I_{w_j}.