Math 978 Homework #1

[Knot theory in $\mathbb{R}^3$ and in S^3 are "the same."]

$S^3 = \mathbb{R}^3 \cup \{\infty\}$ = one-point compactification of $\mathbb{R}^3$

S^3 is a manifold: every point has a nbhd $\cong \mathbb{R}^3$.

1. Show that S^3 is connected.

2. Show that if $x, y \in S^3$ then $\underbrace{\exists \text{ homeo } h: S^3 \to S^3 \text{ with } h(x) = y}_{(*)}$.

[Hint: Show that $(*)$ is an equivalence relation, whose equiv. classes are open subsets of S^3.]

3. Show that any homeo $h: \mathbb{R}^3 \to \mathbb{R}^3$ extends to a homeo $\overline{h}: S^3 \to S^3$

4. Suppose $x, y \in S^3$ and $K \subseteq S^3$ is a knot with $K \cap \{x, y\} = \emptyset$. Show that $S^3 \setminus \{x\}$ and $S^3 \setminus \{y\}$ are both $\cong \mathbb{R}^3$ (via homeos h_1, h_2, say) and, setting $K_1 = h_1(K)$, $K_2 = h_2(K)$, K_1 and K_2 are (unoriented) equivalent.

5. Show that if $K_1, K_2 \subseteq \mathbb{R}^3 \subseteq S^3$ are knots, then $\exists$ homeo $h: \mathbb{R}^3 \to \mathbb{R}^3$ with $h(K_1) = K_2 \iff \exists$ homeo $h': S^3 \to S^3$ with $h'(K_1) = K_2$.