# Math 971 Algebraic Topology

April 26, 2005

Invariance of Domain in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of $f$ with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.

This also gives the more elementary: if $\mathbb{R}^n \cong \mathbb{R}^m$, via $h$, then $n = m$ . Another proof: by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\widetilde{H}_i(S^{n-1}) \cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0)$$
$$\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1})$$

Setting $i = n-1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$ .

**Homology and homotopy groups:** There are connections between homology groups and the fundamental (and higher) homotopy groups, provided by what is known as the *Hurewicz map* $H : \pi_n(X, x_0) \to H_n(X)$ . or $n = 1$ (higher $n$ are similar) the idea is that elements of $\pi_1(X)$ are loops, which can be thought of as maps $\gamma : S^1 \to X$ (or more precisely, mapping into the path component containing $x_0$), inducing a map $\gamma_* : \mathbb{Z} = H_1(S^1) \to H_1(X)$ . We define $H([\gamma]) = \gamma_*(1)$ . Because homotopic maps give the same induced map on homology, this really is well-defined map on homotopy classes, i.e. from $\pi_1(X)$ to $H_1(X)$. [A different view: a loop $\gamma : (I, \partial I) \to (X, x_0)$ defines a singular 1-chain which, being a loop, has zero boundary, so is a 1-cycle. Since based homotopic maps give homologous chains (essentially by the same homotopy invariance property above), we get a well-defined map $\pi_1(X, x_0) \to H_1(X)$.
Since as 1-chains, the concatenation $\gamma * \delta$ of two loops is homologous to the sum $\gamma + \delta$ - the map $K : I \times I \to X$ given by $K(s, t) = (\gamma * \delta)(s)$, after crushing the left and right vertical boundaries to points, can be thought of as a singular 2-simplex with boundary $\gamma + \delta - (\gamma * \delta)$ - the map $H$ is a homomorphism.

When $X$ is path-connected, this map $H : \pi_1(X) \to H_1(X)$ is onto . [When it isn't it maps onto the summand of $H_1(X)$ corresponding to the path component containing our chosen basepoint.] To see this, note that any cycle $z \in Z_1(X)$ can be represented as a sum of singular 1-simplices $\sum \sigma_i^1$ , i.e. we can (by reversing the orientations on simplices to make coefficient positive, and then writing a multiple of a simplex as a sum of simplices) assume all coefficients in our sum are 1. Then $0 = \partial z = \sum(\sigma_i^1(0,1) - \sigma_i^1(1,0))$ means that, starting with any positive term, we can match it with a negative term to cancel that term, which is paired with a postive term, having a matching negative term, etc., until the initial positive term is cancelled. This sub-chain represents a collection of paths which concatenate to a loop, so $z =$ (this loop) + (the remainiung terms) . Induction implies that $z$ can be written as a sum of (sums of paths forming loops), which is (as above) homologous to the sum of loops. Choosing paths from the start of these loops to our chosen basepoint (which is the only place where we use path connectedness, we can concatenate the based loops $\overline{\gamma} * \sigma * \gamma$ to a single based loop $\eta$, which under $H$ is sent to a chain homologous to $z$. So $H[\eta] = [z]$ .

Since $H_1(X)$ is abelian (and $\pi_1(X)$ need not be), the kernel of $H$ contains the commutator subgroup $[\pi_1(X), \pi_1(X)]$ . We now show that, if $X$ is path connected, $H$ induces an isomorphism $H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$ . To show this, it remains to show that $\ker(H) \subseteq [\pi_1(X), \pi_1(X)]$ . Or put differently, the ineduced map from $\pi_1(X)_{ab} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ (i.e., $\pi_1(X)$, written using additive notation) to $H_1(X)$ is injective. So suppose $[\gamma] \in \pi_1(X)$ and, thought of as a singular 1-simplex, $\gamma = \partial w$ for some 2-simplex $w = \sum a_i\sigma_i^2$ . As before, we may assume that all $a_i = 1$, by reversing orientation and writing multiples as sums. By adding "tails" from each image of a vertex of each $\sigma_i^2$ to our chosen basepoint $x_0$, we may assume that the image of every face of $\Delta^2$, under the $\sigma_i$ , is a loop at $x_0$ (by essentially replacing each $\sigma_i$ with a $\tau_i$ which first collapses little triangle at each vertex to arcs, maps the resulting central triangle via $\sigma_i$, and the arcs via the paths).

Once we have made this slight alteration, the equation $\gamma = \partial w = \sum_{i=1}^{n} \sum_{j=0}^{2} \partial_j \sigma_i = 0$ makes sense (and is true) in both ($C_1(X)$ hence $Z_1(X)$ hence) $H_1(X)$ and $\pi_1(X)_{ab}$, the first essentially by definition and the second because all of the $\partial_j \sigma_i$ are loops at $x_0$ and, in $\pi_1(X)$, $(\partial_0 \sigma_i)\overline{\partial_1 \sigma_i}(\partial_2 \sigma_i)$ is null-homotopic, so is trivial in $\pi_1(X)$. Written additively, this means that in $\pi_1(X)_{ab}$ , $\partial_0 \sigma_i - \partial_1 \sigma_i + \partial_2 \sigma_i = 0$. So $\gamma = 0$ in $\pi_1(X)_{ab}$ , as desired.