Math 971 Algebraic Topology

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The isomorphism between simplicial and singular homology provides very quick proofs of several results about singular homology, which would other would require some effort:

If the Δ-complex X has no simplices in dimension greater than n, then $H_i(X) = 0$ for all $i > n$.

This is because the simplicial chain groups $C_i^\Delta(X)$ are 0, so $H_i^\Delta(X) = 0$.

If for each n, the Δ-complex X has finitely many n-simplices, then $H_n(X)$ is finitely generated for every n.

This is because the simplicial chain groups $C_n^\Delta(X)$ are all finitely generated, so $H_n^\Delta(X)$, being a quotient of a subgroup, is also finitely generated. [We are using here that the number of generators of a subgroup H of an *abelian* group G is no larger than that for G; this is not true for groups in general!]

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A, B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex K whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to K, with boundary a (for the exact same reasons as above) closed orientable surface.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

We will approach this through the **Brouwer-Jordan Separation Theorem:** an embedded $(n-1)$-sphere in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two path components. And for this we need to do a slightly unusual homology calculation:

For $k < n$ and $h : I^k \to S^n$ an embedding of a k-cube in to the n-sphere, $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$ for all i.

Here $I = [-1, 1]$. The proof proceeds by induction on k. For $k = 0$, $S^n \setminus h(I^k) \cong \mathbb{R}^n$, and the result follows. Now suppose the result os true for all embeddings of $C = I^{k-1}$, but is false for some embedding $h : I^k \to S^n$ and some i. Then if we divide the cube along its last coordinate, say, as $I^{k-1} \times [-1, 0] = C \times [-1, 0]$ and $C \times [0, 1]$, we can set $A = S^n \setminus h(C \times [-1, 0])$, $B = S^n \setminus h(C \times [0, 1])$, $A \cup B = S^n \setminus h(C \times \{0\})$, and $A \cap B = S^n \setminus h(I^k)$. These sets are all open, since the image under h of the various sets is compact, hence closed. By hypothesis, $A \cup B = S^n \setminus h(C \times \{0\})$ has trivial reduced homology, while $A \cap B = S^n \setminus h(I^k)$ has non-trivial reduced homology in some dimension i. Then the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_{i+1}(A \cup B) \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \widetilde{H}_i(A \cup B) \to \cdots$$

reads $0 \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to 0$ so $\widetilde{H}_i(A \cap B) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$, so at least one of the groups on the right must be non-trivial, as well. WOLOG $\widetilde{H}_i(B) = \widetilde{H}(S^n \setminus h(C \times [0, 1])) \neq 0$. Even more, choosing (once and for all) a non-zero element $[z] \in \widetilde{H}_I(A \cap B)$, snce its image in the direct sum is non-zero, it's coordinate in (say) $\widetilde{H}_i(B)$ is non-zero.