Math 971 Algebraic Topology

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There is another piece of homological algebra that we will find useful ; the Five Lemma. It allows us to compare the information contained in two long exact sequences.

Five Lemma: If we have abelian groups and maps

$$\begin{array}{ccccccccc} A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & D_n & \xrightarrow{i_n} & E_n \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \xrightarrow{h_{n-1}} & D_{n-1} & \xrightarrow{i_{n-1}} & E_{n-1} \end{array}$$

where the rows are exact, and the maps $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then γ is an isomorphism.

The proof is in some sense literally a matter of doing the only thing you can. To show injectivity, suppose $x \in C_n$ and $\gamma x = 0$, then $h_{n-1}\gamma x = \delta h_n x = 0$, so, since δ is injective, $h_n x = 0$. So by the exactness at C_n, $x = g_n y$ for some $y \in B_n$. Then $g_{n-1}\beta y = \gamma g_n y = \gamma x = 0$, so by exactness at B_{n-1}, $\beta y = f_{n-1}z$ for some $z \in A_{n-1}$. Then since α is surjective, $f_{n-1}z = \alpha w$ for some w. Then $0 = g_n f_n w$. But $\beta f_n w = f_{n-1}\alpha w\ f_{n-1}z = \beta y$, so since β is injective, $y = f_n w$. So $0 = g_n f_n w = g_n y = x$. So $x = 0$. For surjectivity, suppose $x \in C_{n-1}$. Then $h_{n-1}x \in D_{n-1}$, so since δ is surjective, $h_{n-1}x = \delta y$ for some $y \in D_n$. Then $\epsilon i_n y = i_{n-1}\delta y = i_{n-1}h_{n-1}x = 0$, so since ϵ is injective, $i_n y = 0$. So by exactness at D_n, $y = h_n z$ for some $z \in C_n$. Then $h_{n-1}\gamma z = \delta h_n z = \delta y = h_{n-1}x$, so $h_{n-1}(\gamma z - x) = 0$, so by exactness at C_{n-1}, $\gamma z - x = g_{n-1}w$ for some $w \in B_{n-1}$. Then since β is surjective, $w = \beta u$ for some $u \in B_n$. Then $\gamma g_n u = g_{n-1}\beta u = g_{n-1}w = \gamma z - x$, so $x = \gamma z - \gamma g_n u = \gamma(z - g_n u)$. So γ is onto.

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\mathrm{cl}_X(B) \subseteq \mathrm{int}_X(A)$, then for every k the inclusion-induced map $H_k(X \setminus B, A \setminus B) \to H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\mathrm{int}_X(A)\cup\ \mathrm{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \to H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$.] To prove the second statement, we know that the inclusion $C_n^{\{A,B\}}(X) \to C_n(X)$ induce isomorphisms on homology, as does $C_n(A) \to C_n(A)$, so, by the five lemma, the induced map $C_n^{\{A,B\}}(X)/C_n(A) \to C_n(X)/C_n(A) = C_n(X, A)$ induces isomorphisms on homology. But the inclusion $C_n(B) \to C_n^{\{A,B\}}(X)$ induces a map $C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \to C_n^{\{A,B\}}(X)/C_n(A)$ which is an isomorphism of chain groups; a basis for $C_n^{\{A,B\}}(X)/C_n(A)$ consists of singular simplices which map into A or B, but don't map into A, i.e., of simplices mapping into B but not A, i.e., of simplices mapping into B but not $A \cap B$. But this is the same as the basis for $C_n(B, A \cap B)$!

With these tools, we can start making some real homology computations. First, we show that if $\emptyset \neq A \subseteq X$ is "nice enough", then $H_n(X, A) \cong \widetilde{H}_n(X/A)$. The definition of nice enough, like Seifert - van Kampen, is that A is closed and has an open neighborhood $\mathcal{U}$ that deformation retracts to A (think: A is the subcomplex of a CW-complex X). Then using $\mathcal{U}.X \setminus A$ as a cover of X, and $\mathcal{U}/A, (X \setminus A)/A$ as a cover of X/A, we have

$$\widetilde{H}_n(X/A) \overset{(1)}{\cong} H_n(X/A, A/A) \overset{(2)}{\cong} H_n(X/A, \mathcal{U}/A) \overset{(3)}{\cong} H_n(X/A \setminus A/A, \mathcal{U}/A \setminus A/A) \overset{(4)}{\cong} H_n(X \setminus A, \mathcal{U} \setminus A) \overset{(5)}{\cong} H_n(X, A)$$

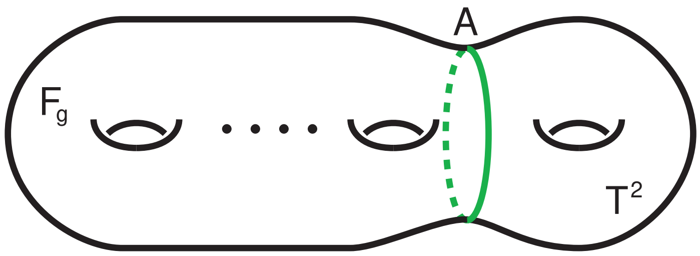
Where (1),(2) follow from the LES for a pair, (3),(5) by excision, and (4) because the restriction of the quotient map $X \to X/A$ gives a homeomorphism of pairs.

Second, if X, Y are T_1, $x \in X$ and $y \in Y$ each have neighborhoods $\mathcal{U}, \mathcal{V}$ which deformation retract to each point, then the one-point union $Z = X \vee Y = (X \coprod Y)/(x = y)$ has $\widetilde{H}_n(Z) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$; this follows from a similar sequence of isomorphisms. Setting $z =$ the image of $\{x, y\}$ in Z, we have

$$\widetilde{H}_n(Z) \cong H_n(Z, z) \cong H_n(Z, \mathcal{U} \vee \mathcal{V}) \cong H_n(Z \setminus z, \mathcal{U} \vee \mathcal{V} \setminus z) \cong H_n([X \setminus x] \textstyle\coprod [Y \setminus y], [\mathcal{U} \setminus x] \coprod [\mathcal{V} \setminus y]) \cong$$
$$H_n(X \setminus x, \mathcal{U} \setminus x) \oplus H_n(Y \setminus y, \mathcal{V} \setminus y) \cong H_n(X, x) \oplus H_n(Y, y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

By induction, we then have $\widetilde{H}_n(\vee_{i=1}^k X_i) \cong \oplus_{i=1}^k \widetilde{H}_n(X_i)$

As an application, we can compute the homology groups of the closed orientable surfaces of genus g, F_g. We will argue by induction on g that $\widetilde{H}_i(F_g) = \mathbb{Z}$ for $i = 2$, $\mathbb{Z}^{2g}$ for $i = 1$, and 0 for all other i. For the base case g=1 we have the 2-torus, whose homology we have computed previously. For the inductive step, we look at the homology of the pair (F_{g+1}, A) depicted here; with what we have learned above, in the LES for the pair, we will have $H_i(F_{g+1}, A) \cong \widetilde{H}_i(F_{g+1}/A) \cong \widetilde{H}_i(F_g \vee T^2) \cong \widetilde{H}_i(F_g) \oplus \widetilde{H}_i(T^2)$, which we can compute, by our inductive hypothesis. And so we find that we have
$\widetilde{H}_2(A) \to \widetilde{H}_2(F_{g+1}) \to \widetilde{H}_2(F_g \oplus \widetilde{H}_2(T^2) \to \widetilde{H}_1(A) \to \widetilde{H}_1(F_{g+1}) \to \widetilde{H}_1(F_g) \oplus \widetilde{H}_2(T^2) \to \widetilde{H}_0(A)$
which renders as $\quad 0 \to \widetilde{H}_2(F_{g+1}) \to \mathbb{Z}^2 \to \mathbb{Z} \to \widetilde{H}_1(F_{g+1}) \to \mathbb{Z}^{2g+2} \to 0 \quad$. The point, though is that the map $\widetilde{H}_1(A) \to \widetilde{H}_1(F_{g+1})$ is the 0 map, since the circle A is null-homologous in F_{g+1}; it bounds the once-punctured torus X on the right of the figure. [Formally, what we mean is that the generator of $\widetilde{H}_1(A)$, a singular 1-simplex wrapping once around A, is a boundary in $C_1(F_{g+1})$; writing X as a sum of singular 2-simplices provides a demonstration.] So we can cut out exact sequence into two pieces
$0 \to \widetilde{H}_2(F_{g+1}) \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \qquad$ and $\qquad 0 \to \widetilde{H}_1(F_{g+1}) \to \mathbb{Z}^{2g+2} \to 0$
The first implies that $\widetilde{H}_2(F_{g+1})$ is a direct summand, with $\mathbb{Z}$, for $\mathbb{Z}^2$, and so is $\mathbb{Z}$; the second asserts that $\widetilde{H}_1(F_{g+1}) \cong \mathbb{Z}^{2g+2}$, as desired. So our inductive step is proved, and our computation follows by induction.



F_{g+1}