Math 971 Algebraic Topology
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Relative homology: we build the singular chain complex of a pair (X, A) , i.e., of a space X and a subspace $A \subseteq X$. Since as abelian groups we can think of $C_n(A)$ as a subgroup of $C_n(X)$ (under the injective homomorphism induced by the inclusion $i : A \to X$) we can set $C_n(X, A) = C_n(X)/C_n(A)$. Since the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ satisfies $\partial_n(C_n(A) \subseteq C_{n-1}(A)$ (the boundary of a map into A maps into A), we get an induced boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. These groups and maps $(C_n(X, A), \partial_n)$ form a chain complex, whose homology groups are the *singluar relative homology groups of the pair* (X, A) . To be a cycle in relative homology, you need to have a representative z with $\partial z \in C_{n-1}(A)$, i.e., you are a chain with boundary in A. To be a boundary, you need $z = \partial w + a$ for some $w \in C_{n+1}(X)$ and $a \in C_n(A)$, i.e., you *cobound* a chain in A ($\partial w = z - a$). Note that the relative homology of the pair $(X, \emptyset)$ is just the ordinary homology of X; we aren't modding out by anything.

The inclusion i_n and projection p_n maps give us short exact sequences

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

and since the boundary on chains in X restricts to the boundary on A and induces the boundary on (X, A), i_n and p_n are chain maps. So we get a long exact homology sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

We can also replace the absolute homology groups in this sequence with reduced homology groups, by augmenting the short exact seuences with $0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$ at the bottom. There is also a long exact sequence of a triple (X, A, B) , where by triple we mean $B \subseteq A \subseteq X$. From the short exact sequences $0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$ (i.e.,

$$0 \to C_n(A)/C_n(B) \to C_n(X)/C_n(B) \to C_n(X)/C_n(A) \to 0)$$

we get the long exact sequence

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to H_{n-1}(X, B) \to \cdots$$

So for example if we look at the pair $(\mathbb{D}^n, \partial\mathbb{D}^n) = (\mathbb{D}^n, S^{n-1})$, since the reduced homology of $\mathbb{D}^n$ is trivial in every dimension, every third group in our LES is 0, giving $H_m(\mathbb{D}^n, S^{n-1}) \cong \widetilde{H}_{m-1}(S^{n-1})$ for every m and n.

A basic fact is that if A sits in X "nicely enough" (think: A is a subcomplex of the cxell complex X), then $H_n(X, A) \cong \widetilde{H}_n(X/A)$. We will shortly prove this! One nice consequence is that we can do some (non-trivial!) basic calculations: taking $X = \mathbb{D}^n$ and $A = \partial\mathbb{D}^n = S^{n-1}$, we have $\mathbb{D}^n/S^{n-1} \cong S^n$, and the previous two facts combine to give $\widetilde{H}_m(S^n) \cong \widetilde{H}_{m-1}(S^{n-1})$ for every m and n . By induction (since we know that values of $\widetilde{H}_{m-n}(S^0)$, we find that $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and all other homology groups are 0.

And this, in turn, let's us prove a fairly sizable theorem:

Brouwer Fixed Point Theorem: For every n, every map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

Proof: If $f(x) \neq x$ for every x, then is with the $n = 2$ case that you may have seen before, we can construct a retraction $r : \mathbb{D}^n \to \partial\mathbb{D}^n = S^{n-1}$ by setting $r(x) =$ the (first) point past $f(x)$ where the ray from $f(x)$ to x meets $\partial\mathbb{D}^n$. This function is continuous, and is the identity on the boundary. So from our of your problem sets, the inclusion-induced map $i_* : H_{n-1}(S^n) \to H_{n-1}(\mathbb{D}^n)$ is injective. But this is impossible, since the first group is $\mathbb{Z}$ and the second is 0 .

Another source of short exact sequences is *homology with coefficients.* In ordinary (singular) homology, our chains are formal linear combinations of singular simplices, with coefficients in $\mathbb{Z}$. But all we needed to know about $\mathbb{Z}$ was that we could add things, and that integers have negatives (and how to recognize 0). So, any abelian group G should work. If we define singular chains with coefficients in G to be formal linear combinations $\sum g_i\sigma_i^n$, then since the boundary map is computed simplex by simplex, we can define $\partial(g\sigma) = \sum(-1)^i g\sigma|_{\Delta_i^{n-1})}$, essentially as before, and get a new chain complex $C_*(X; G)$. It's homology groups (cycles/boundaries) is the *homology of X with coefficients in G*, $H_*(X; G)$. We can also define relative homology groups $H_*(X, A; G)$ in exactly the same way as before.

From this perspective, our original homology groups $H_n(X)$ should be called $H_n(X;\mathbb{Z})$. And the point, in the context of our present discussion, is that a short exact sequence of coefficient groups, $0 \to K \to G \to H \to 0$ induces a short exact sequence of chain groups $0 \to C_n(X;K) \to C_n(X;G) \to C_n(X;H) \to 0$, giving us a long exact homology sequence

$$\cdots \to H_{n+1}(X;H) \to H_n(X;K) \to H_n(X;G) \to H_n(X;H) \to H_{n-1}(X,K) \to \cdots$$

So for example, the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$, where the first map is multiplication by n, and the second is reduction mod n, is exact, and gives us a long exact sequence involving ordinary homology and homology mod n. Everything we have done with homology so far goes through with coefficients, essentially with the identical proof; foi example, homotopy equivalent spaces have isomorphic homology with coefficients, and homotopic maps induce the same maps on homology.