Math 971 Algebraic Topology

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Some examples: the Klein bottle K has a Δ-complex structure with 2 2-simplices, 3 1-simplices, and 1 0-simplex; we will call them $f_1 = [0,1,2], f_2 = [1,2,3], e_1 = [0,2] = [1,3], e_2 = [1,0] = [2,3], e_3 = [1,2]$, and $v_1 = [0] = [1] = [2] = [3]$. Computing, we find $\partial_2 f_1 = \partial[0,1,2] = [1,2] - [0,2] + [0,1] = e_3 - e_1 - e_2$, $\partial_2 f_2 = e_2 - e_1 + e_3$, $\partial_1 e_1 = \partial_1 e_2 = \partial_1 e_3 = 0$ and $\partial_i = 0$ for all other i (as well). So we have the chain complex

$\cdots \to 0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$

and all of the boundary maps are 0, except for ∂_2, which has the matrix $\begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$. This matrix is injective, so $\ker \partial_2 = 0$, so $H_2(K) = 0$, on the other hand, $H_1(K) = \mathrm{coker}(\partial_2)$, and applying column operations we can transform the matrix for ∂_2 to $\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 0 \end{pmatrix}$, which implies that the cokernel is $\mathbb{Z} \oplus \mathbb{Z}_2$, since $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis for $\mathbb{Z}^3$. Finally, $H_0(K) = \mathbb{Z}$, since $\partial_1, \partial_0 = 0$, and all higher homology groups are also 0, for the same reason.

As another example, the topologist's dunce hat has a Δ-structure with 1 2-simplex, 1 1-simplex, and 1 0-simplex. The boundary maps, we can work out (starting from $C_2(X)$), are $(1), (0)$, and (0), so $H_2(X) = H_1(X) = 0$, and $H_0(X) = \mathbb{Z}$. all higher groups are also 0.

These homology groups are, in the end, fairly routine to calculate from a Δ-complex structure. But there is one very large problem; the calculations depend on the Δ structure! This is not a group defined from the space X; it is defined from the space and a Δ structure on it. A priori, we don't know that if we chose a different structure on the same space, that we would get isomorphic groups! We should really denote our groups by $H_i^{\Delta}(X)$, to acknowledge this dependence on the structure.

But we don't *want* a group that depends on this structure. We want groups that just depend on the topological space X, i.e., which are topological invariants. In really turns out that these groups $H_i^{\Delta}(X)$ are topological invariants, but we will need to take a very roundabout route to show this. What we will do now is to define another sequence $H_i(X)$ of groups, the *singular homology groups*, which their definition makes apparent from the outset that they are topological invariants. But this definition will also make it very unclear how to really compute them! Then we will show that for Δ-complexes these two sequences of groups are really the same. In so doing, we will have built a sequence of topological invariants that for a large class of spaces are fairly routine to compute. Then all we will need to show is that they also capture useful information about a space (i.e., we can prove useful theorems with them!).

And the basic idea behind defining them is that, with simplicial homology, we have already done all of the hard work. What we do is, as before, build a sequence of (free) abelian groups, the chain groups $C_n(X)$, and boundary maps between them, with consecutive maps composing to 0. Then, as before, the homology groups are kernels mod images, i.e., cycles mod boundaries. And, as before, the basis elements for each of our chain groups $C_n(X)$ will be the n-simplices in X. But now X is any topological space. So how do we get n-simplices in such a space? We do the only thing we can; we *map* them in.

More precisely, we work with *singular n-chains*, that is, formal (finite) linear combinations $\sum a_i\sigma_i$, where $a_i \in BbbZ$ and the σ_i are *singular simplices*, that is, (continuous) maps $\sigma_i : \Delta^n \to X$ from the (standard) n-simplex into X. The boundary maps are really exactly as before; they are the alternating sum of the restrictions of σ_i to the $n+1$ faces of Δ^n . (Formally, we must precompose these face maps with the (orientation-preserving) linear isomorphism from the standard $(n-1)$-simplex to each of the faces, preserving the ordering of their vertices.) The same proof as before (except that we interpret the faces as restrictions of the map σ_i, instead of as physical faces) shows that the composition of two successive boundaries are 0, and so all of the machinery is in place to define the *singular homology groups* $H_i(X)$ as the kernel of ∂_i modulo the image of $\partial_{i+1} = Z_i(X)/B_i(X)$. They are, by their definition, groups defined using the topological space X as input, and so are topological invariants of X. The elements are equivalence classes of i-cycles, where $z_1 \sim z_2$ if $z_1 - z_2 = \partial w$ for some $(i+1)$-chain w . We say that z_1 and z_2 are *homologous*.