Math 971 Algebraic Topology
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We typically think of building a Δ-complex X inductively. The *0-simplices* (i.e., points), or *vertices*, form the 0-skeleton $X^{(0)}$. n-simplices $\sigma^n = [v_0, \dots v_n]$ attach to the $(n-1)$-skeleton to form the n-skeleton $X^{(n)}$; the restriction of the attaching map to each face of σ^n is, by definition, an $(n-1)$-simplex in X. The attaching map is (by induction) really determined by a map $\{v_0, \dots, v_n\} \to X^{(0)}$, since this determines the attaching maps for the 1-simplices in the boundary of the n-simplex, which gives 1-simplices in X, which then give the attaching maps for the 2-simplices in the boundary, etc. Note that the reverse is not true; the vertices of two different n-simplices in X can be the same. For example, think of the 2-sphere as a pair of 2-simplices whose boundaries are glued by the identity. Δ-complexes generalize *simplicial complexes* where the simplices are required to attach by homeomorphisms to the skeleton, and the intersection of two simplices are a (single) sub-simplex of each. This has the advantage over Δ-complexes that an n-simplex is determined uniquely by the set of vertices in $X^{(0)}$ that it attaches to. This means that, in principle, a simplicial complex (and everything associated with it, e.g., its homology groups!) can be treated purely combinatorially; the complex is "really" a certain collection of subsets of the vertices (since these determine the simplices), with the property that any subset B of a subset A that has been declared to be a simplex is also a simplex. But they have the disadvantage that it typically takes far more simplices to build a simplicial structure on a space X that it does to build a Δ-structure. This makes the computations we are about to do take far longer.

The final detail that we need before defining (simplicial) homology groups is the notion of an *orientation* on a simplex of X. Each simplex σ^n is determined by a map $f : \{v_0, \dots, v_n\} \to X^{(0)}$; an orientation on σ^n is an (equivalence class of) the ordered $(n+1)$-tuple $(f(v_0), \dots f(v_n)) = (V_0, \dots, V_n)$. Another ordering of the same vertices represents the same orientation if there is an *even* permutation taking the entries of the first $(n+1)$-tuple to the second. This should be thought of as a generalization of the right-hand rule for $\mathbb{R}^3$, interpreted as orienting the vertices of a 3-simplex. Note that there are precisely two orientations on a simplex.

Now to define homology! We start by defining *n-chains*; these are (finite) formal linear combinations of the (oriented!) n-simplices of X, where $-\sigma$ is interpreted as σ with the opposite (i.e., other) orientation. Adding formal linear combinations formally, we get the n-th *chain group* $C_n(X) = \{\sum n_\alpha \sigma_\alpha : \sigma_\alpha$ an oriented n-simplex in $X\}$. We next define a *boundary operator* $\partial : C_n(X) \to C_{n-1}(X)$, whose image will be the $(n-1)$-chains that are the "boundaries" of n-chains. The idea is that the boundary of a 2-simplex, for example, should be a "sum" of its three faces (since they do make up the boundary of the simplex), but we need to take into account their orientations, in order to be getting the correct sum. Thinking of the orientation on a 1-simplex $[v, w]$ as an arrow pointing from v to w, we are lead to believe that the boundary of a 2-simplex $[u, v, w]$ should be $[u, v] + [v, w] + [w, u]$. Similarly, the boundary of $[u, v]$, on reflection, should be $[v] - [u]$, to distinguish the head of the arrow (the $+$ side) from the tail (the $-$ side). On the basis of these examples, trying to find a consistent formula, one might eventually be led to the following formulation. We define the boundary on the basis elements $\sigma_\alpha = \sigma$ of $C_n(X)$ as $\partial\sigma = \sum (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}$, where $\sigma : [v_0, \dots, v_n] \to X$ is the characteristic map of σ_α . $\partial\sigma$ is therefore an alternating sum of the faces of σ. We then extend the definition by linearity to all of $C_n(X)$. When a notation indicating dimension is needed, we write $\partial = \partial_n$. We define $\partial_0 = 0$.

This definition, it utrns out, is cooked up to make the maxim "boundaries have no boundary" true; that is, $\delta_{n-1} \circ \delta_n = 0$, the 0 map. This is because, for any simplex $\sigma = [v_0, \dots v_n]$,

$$\delta \circ \delta(\sigma) = \delta\Big(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}\Big)$$
$$= \Big(\sum_{j<i} (-1)^j (-1)^i \sigma|_{[v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_n]}\Big) + \Big(\sum_{j>i} (-1)^{j-1} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]}\Big)$$

The distinction between the two pieces is that in the second part, v_j is actually the $(j-1)$-st vertex of the face. Switching the roles of i and j in the second sum, we find that the two are negatives of one another, so they sum to 0, as desired.

And this little calculation is all that it takes to define homology groups! What this tells us is that $\text{im}(\delta_{n+1}) \subseteq \ker(\delta_n)$ for every n. $\ker(\delta_n) = Z_n(X)$ are called the *n-cycles* of X; they are the n-chains with 0 (i.e., empty) boundary. They form a (free) abelian subgroup of $C_n(X)$. $\text{im}(\delta_{n+1}) = B_n(X)$ are the *n-boundaries* of X; they are, of course, the boundaries of $(n+1)$-chains in X. The n-th homology group of X, $H_n(X)$ is the quotient $Z_n(X)/B_n(X)$; it is an abelian group. This consists, essentially, of the n-cycles that are not boundaries; i.e., they represent the "holes" in X.

A key observation is that the boundary maps δ_n are linear, that is, they are homomorphisms between the free abelian groups $\delta_n : C_n(X) \to C_{n-1}(X)$. Consequently, they can be expressed as (integer-valued) matrices Δ_n. Row reducing Δ_n (over the integers!) allows us to find a basis $v_1, \ldots, v_k$ for $Z_n(X)$ (clearing denomenators to get vectors over $\mathbb{Z}$). Then since $\Delta_n \Delta_{n+1} = 0$, the columns of Δ_{n+1} are in the kernel of Δ_n, so can be expressed as linear combinations of the v_i . These combinations can be determined by row reducing the augmented matrix $(v_1 \cdots v_k | \Delta_{n+1})$. This will row reduce to $\begin{pmatrix} I & | & C \\ 0 & | & 0 \end{pmatrix}$, and C basically describes the boundaries $B_n(X)$ in terms of the basis $v_1, \ldots, v_k$. The homology group $H_n(X)$ is then the *cokernel* of C, i.e., $\mathbb{Z}^k/\text{im}C$. Note that C will have integer entries, since we know that the columns of Δ_{n+1} can be expressed as integer linear combinations of the v_i, and, being a basis, there is only one such expression.