Math 971 Algebraic Topology

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Building universal coverings: If a space X is path connected, locally path connected, and semi-locally simply connected (S-LSC), then it has a universal covering; we describe a general construction. The idea is that a covering space should have the path lifting and homotopy lifting properties, and the universal cover can be characterized as the only covering space for which *only* null-homotopic loops lift to loops. So we build a space and a map which <u>must</u> have these properties. We do this by making a space $\widetilde{X}$ whose points are (equivalence classes of) $[\gamma]$ based paths $\gamma : (I, 0) \to (X, x_0)$, where two paths are equivalent if they are homotopic rel endpoints! The projection map is $p([\gamma]) = \gamma(1)$. The S-LSCness is what guarantees that this is a covering map; choosing $x \in X$, a path γ_0 from x_0 to x, and a neighborhood $\mathcal{U}$ of x guaranteed by S-LSC, paths from x_0 to points in $\mathcal{U}$ are based equivalent to $\gamma * \gamma_0 * \eta$ where γ is a based loop at x_0 and η is a path in $\mathcal{U}$. But by simple connectivity, a path in $\mathcal{U}$ is determined up to homotopy by its endpoints, and so, with γ fixed, these paths are in one-to-one correspondence with $\mathcal{U}$. So $p^{-1}(\mathcal{U}$ is a disjoint union, indexed by $\pi_1(X, x_0)$, of sets in bijective correspondence with $\mathcal{U}$. The appropriate topology on $\widetilde{X}$, essentially given as a basis by triples $\gamma *, gamma_0, \mathcal{U}$ as above, make p a covering map. Note that the inverse image of the basepoint x_0 is the equivalence classes of <u>loops</u> at x_0, i.e., $\pi_1(X, x_0)$. A path γ lifts to the path of paths $[\gamma_t]$, where $\gamma_t(s) = \gamma(ts)$, and so the only loop in X which lifts to a loop in $\widetilde{X}$ has endpoint $[\gamma] = [c_{x_0}]$, i.e., $[\gamma] = 1$ in $\pi_1(X, x_0)$. This implies that $p_*(\pi_1(\widetilde{X}, [c_{x_0}])) = \{1\}$, so $\pi_1(\widetilde{X}, [c_{x_0}]) = \{1\}$. However, nobody in their right minds would go about building $\widetilde{X}$ in this way, in general! Before describing how to do it "right", though, we should perhaps see why we should want to?

One reason for the importance of the universal cover is that it gives us a unified approach to building <u>all</u> connected covering spaces of X. The basis for this is the *deck transformation group* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$. These homeomorphisms, by definition, permute each of the point inverses of p. In fact, since h can be thought of as a lift of the projection p, by the lifting criterion h is determined by which point in the inverse image of the basepoint x_0 it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give the map h, and the opposite inclusion to ensure it is a bijection (because its inverse exists)]. These two groups are in general *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$; this can be seen by following the change of basepoint isomorphism down into $G = \pi_1(X, x_0)$. As we have seen, paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 correspondence with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $p_*(\pi_1(X, x_0)$; so deck transformations are in 1-to-1 correspondence with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of H in G*, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore in 1-to-1 correspondence with the group $N(H)/H$ under $h \mapsto$ the coset represented by the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$. And since h is essentially built by lifting paths, it follows quickly that this map is a homomorphism, hence an isomorphism.

In particular, applying this to the universal covering space $p : \widetilde{X} \to X$, since in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$, its deck transformation group is isomorphic to $\pi_1(X, x_0)$. For example, this gives the quickest possible proof that $\pi_1(S^1) \cong \mathbb{Z}$, since $\mathbb{R}$ is a contractible covering space, whose deck transformations are the translations by integer distances. Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action <u>is</u> the projection map p. The evenly covered property of p implies that X does have the quotient topology under this action.

So every space it X the quotient of its universal cover (if it has one!) by its fundamental group $G = \pi_1(X, x_0)$, realized as the group of deck transformations. And the quotient map is the covering projection.