Math 970 mid-semester review

Set-theoretic beginnings:

Functions: $f : X \to Y$. injection, surjection, bijection; image,
inverse image $f^{-1}(A) = \{x \in X : f(x) \in A\}$

Image: $f(\bigcup A_\alpha) = \bigcup f(A_\alpha)$, $f(\bigcap A_\alpha) \subseteq \bigcap f(A_\alpha)$

Inverse image: $f^{-1}(\cup A_\alpha) = \cup f(A_\alpha)$, $f^{-1}(\cap A_\alpha) = \cap f^{-1}(A_\alpha)$, $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Finite sets, infinite sets, countable sets

A is finite $\Leftrightarrow$ $\exists$ a surjection $\{1, \dots, n\} \to A \Leftrightarrow$ $\exists$ an injection $A \to \{1, \dots, n\}$.

A is countable $\Leftrightarrow$ $\exists$ a surjection $\mathbb{N} \to A \Leftrightarrow$ $\exists$ an injection $A \to \mathbb{N}$.

countable union of countables is countable, product of two countables is countable.

Cardinality: $|A| = |B|$ if $\exists$ a bijection $f : A \to B$

Shroeder-Bernstein Thm: if $\exists$ injection $A \to B$ and $\exists$ injection $B \to A$, then $|A| = |B|$

Topologies

Idea: extend continuity to more general settings.

Metric spaces: (X, d) , $d : X \times X \to \mathbb{R}$ satisfies

$d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow$ $x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$.

$f : (X, d) \to (Y, d')$ continuous ($=$ cts) if

$\forall$ $a \in X$ and $\forall$ $\epsilon > 0$ $\exists$ $\delta = \delta(a, \epsilon) > 0$ so that $d(a, x) < \delta \Rightarrow d'(f(a), f(x)) > \epsilon$.

(Open) neighborhood: $N_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$

Open set: $\mathcal{U} \subseteq X$ is open if $\forall$ $x \in \mathcal{U}$ $\exists$ $\epsilon > 0$ so that $N_d(x, \epsilon) \subseteq \mathcal{U}$

$\mathcal{U} \subseteq X$ is open $\Leftrightarrow$ $\mathcal{U} =$ a union of neighborhoods.

$f : X \to Y$ is cts $\Leftrightarrow$ $f^{-1}\mathcal{U}$ is open in X $\forall$ $\mathcal{U}$ open in Y

The collection $\mathcal{T}$ of open sets in (X, d) satisfies

$\emptyset, X \in \mathcal{T}$

if $\mathcal{U}, \mathcal{V} \in \mathcal{T}$, then $\mathcal{U} \cap \mathcal{V} \in \mathcal{T}$

if $\mathcal{U}_\alpha \in \mathcal{T}$ $\forall$ $\alpha \in I$, then $\bigcap \mathcal{U}_\alpha \in \mathcal{T}$

For X any set, a **topology** on X is any collection $\mathcal{T}$ of subsets of X satisfying the above three conditions.

$f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is cts $\Leftrightarrow$ $f^{-1}(\mathcal{U}) \in \mathcal{T}$ for all $\mathcal{U} \in \mathcal{T}'$

comparing topologies: $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}$ is *coarser* than $\mathcal{T}'$; $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

$\mathcal{T} = \mathcal{T}' \Leftrightarrow$ $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}' \subseteq \mathcal{T}$

Examples:

$\mathcal{T}_i = \{\emptyset, X\} =$ trivial topology ($=$ indiscrete topology).

$\mathcal{T}_d = \mathcal{P}(X) =$ all subsets of $X =$ discrete topology.

$\mathcal{T} =$ open sets for a metric d on $X =$ metric topology on X.

$(X, \mathcal{T})$ is *metrizable* if $\mathcal{T}$ is the metric topology for some metric on X.

$\mathcal{T} = \{\mathcal{U} \in X : X \setminus \mathcal{U}$ is finite$\} \cup \{\emptyset\} =$ finite complement topology.

$\mathcal{T} = \{\mathcal{U} \in X : X \setminus \mathcal{U}$ is countable$\} \cup \{\emptyset\} =$ countable complement topology.

For $a \in X$, $\mathcal{T} = \{\mathcal{U} \subseteq X : a \in \mathcal{U}\} \cup \{\emptyset\} =$ included point topology.

For $a \in X$, $\mathcal{T} = \{\mathcal{U} \subseteq X : a \notin \mathcal{U}\} \cup \{X\} =$ excluded point topology.

On $\mathbb{R}$, $\mathcal{T} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} =$ infinite open ray topology.

On $\mathbb{R}$, $\mathcal{T} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} =$ infinite ray topology.

$f : X \to (Y, \mathcal{T}')$, then $\mathcal{T} = \{f^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{T}'\}$ = coarsest top. on X making f cts.
$f : (X, \mathcal{T}) \to Y$, then $\mathcal{T}' = \{\mathcal{U} : f^{-1}(\mathcal{U}) \in \mathcal{T}\}$ = finest topology on Y making f cts.

Metric topologies also satisfy: if $x, y \in X$, $x \neq y$, then $\exists\, \mathcal{U}, \mathcal{V}$ open with $x \in \mathcal{U}, y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$ A topological space satisfying this property is called *Hausdorff*.

A *topological property* is a property which can be described in terms of open sets and relations between them. (For example, Hausdorffness.) Topology is, essentially, the study of topological properties and the relationships between them.

Bases and subbases

Open sets for metric spaces were defined as unions of neighborhoods (= nbhds); this gives a topology because:

X = union of nbhds, and the intersection of two nbhds is a union of nbhds.

A collection $\mathcal{B}$ of subsets of X is a *basis* if it satisfies those two properties, i.e.:

$X = \bigcup\{B : B \in \mathcal{B}\}$, and

if $B, B' \in \mathcal{B}$ and $x \in B \cap B'$, then $\exists\, B'' \in \mathcal{B}$ with $x \in B'' \subseteq B \cap B'$.

The topology $\mathcal{T}(\mathcal{B})$ that it *generates* is the unions of elements of $\mathcal{B}$.

A *subbasis* is a collection $\mathcal{S}$ of subsets whose union is X.

The basis $\mathcal{B}(\mathcal{S})$ that it generates is the set of all finite intersections of elements of $\mathcal{S}$.

$f : (X, \mathcal{T}) \to (Y, \mathcal{T}(\mathcal{B}))$ is cts $\Leftrightarrow$ $f^{-1}(B) \in \mathcal{T}$ for all $B \in \mathcal{B}$
$f : (X, \mathcal{T}) \to (Y, \mathcal{T}(\mathcal{B}(\mathcal{S})))$ is cts $\Leftrightarrow$ $f^{-1}(S) \in \mathcal{T}$ for all $S \in \mathcal{S}$

$\mathcal{U} \in \mathcal{T}(\mathcal{B}) \Leftrightarrow \forall\, x \in \mathcal{U}\ \exists\, B \in \mathcal{B}$ so that $x \in B \subseteq \mathcal{U}$; $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{T} \Leftrightarrow \mathcal{B} \subseteq \mathcal{T}$

On $\mathbb{R}$, $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\}$ is a basis for the usual (metric) topology.
$\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ is also a basis; $\mathbb{R}$ with this topology is called the *Sorgenfrey line*.

New spaces from old

Basic idea: topologies on new sets should be defined to make reasonable functions cts.

$A \subseteq X$, $(X, \mathcal{T})$, then would like $i : A \to X$ continuous, so define
$\mathcal{T}_A = \{i^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{T}\} = \{\mathcal{U} \cap A : \mathcal{U} \in \mathcal{T}\}$ = subspace topology

if $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\{B \cap A : B \in \mathcal{B}\} = \mathcal{B}_A$ is a basis for $\mathcal{T}_A$

If $f : X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous; $f|_A = f \circ i$

If $B \subseteq A \subseteq X$ then the subspace topology B gets from $(A, \mathcal{T}_A)$ is the same as it gets from $(X, \mathcal{T})$.

$(X, \mathcal{T}), (Y, \mathcal{T}')$ top spaces, would like $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ (coord projections) to be cts.

So need $p_X^1(\mathcal{U}) = \mathcal{U} \times Y$ and $p_Y^1(\mathcal{V}) = X \times \mathcal{V}$ open. These form a subbasis, with basis $\mathcal{B} = \{\mathcal{U} \times \mathcal{V} : \mathcal{U} \in \mathcal{T}, \mathcal{V} \in \mathcal{T}'\}$; $\mathcal{T}(\mathcal{B})$ = the *product topology* on $X \times Y = \mathcal{T} \times \mathcal{T}'$.

$f : Z \to X \times Y$ is cts $\Leftrightarrow$ $p_X \circ f$ and $p_Y \circ f$ are both continuous

If $\mathcal{T} = \mathcal{T}(\mathcal{B}), \mathcal{T}' = \mathcal{T}(\mathcal{B}')$, then $\{B \times B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$ is a basis for $\mathcal{T} \times \mathcal{T}'$.

If $A \subseteq X, B \subseteq Y$, then the subspace topology on $A \times B \subseteq X \times Y$ is the same as $\mathcal{T}_A \times \mathcal{T}_B$

Products in general:

$(X_\alpha, \mathcal{T}_\alpha)$ top. spaces, $\alpha \in I$, then there are two reasonable topologies on $\prod X_\alpha$:

box topology: basis is $\{\prod \mathcal{U}_\alpha : \mathcal{U}_\alpha \in \mathcal{T}_\alpha\}$
product topology: subbasis is $\bigcup\{p_\alpha^{-1}(\mathcal{U}_\alpha) : \mathcal{U}_\alpha \in \mathcal{T}_\alpha\}$; p_α = proj to X_α

In the product topology, $f : Z \to \prod X_\alpha$ is cts $\Leftrightarrow$ $p_\alpha \circ f$ is cts for all α

Closed sets

$C \subseteq X$ is closed if $X \setminus C = \mathcal{U} \in \mathcal{T}$; i.e., C is closed if $C = X \setminus \mathcal{U}$ for some $\mathcal{U} \in \mathcal{T}$

$\emptyset, X$ are closed ; C, D closed $\Rightarrow C \cup D$ closed ; C_α closed $\Rightarrow \bigcap C_\alpha$ closed.

$f : X \to Y$ is cts $\Leftrightarrow$ $f^{-1}\mathcal{U}$ is closed in X $\forall$ $\mathcal{U}$ closed in Y

$D \subseteq A \subseteq X$ is closed in $(A, \mathcal{T}_A)$ $\Leftrightarrow$ $C = D \cap A$ for some D closed in $(X, \mathcal{T})$

Closure: $\mathrm{cl}(A) = \overline{A} = \bigcap\{C \subseteq X \text{ closed} : A \subseteq C\}$ = smallest closed set containing A

Interior: $\mathrm{int}(A) = \bigcup\{\mathcal{T} \in \mathcal{T} : \mathcal{U} \subseteq A\}$ = largest open set contained in A

$\mathrm{cl}(X \setminus A) = X \backslash \mathrm{int}(A)$; $\mathrm{int}(X \setminus A) = X \backslash \mathrm{cl}(A)$.

C closed and $A \subseteq C \Rightarrow \overline{A} \subseteq C$

$A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; $\overline{A \cup B} = \overline{A} \cup \overline{B}$

A is closed $\Leftrightarrow$ $\overline{A} = A$; A is open $\Leftrightarrow$ $\mathrm{int}(A) = A$

The closure of $B \subseteq A$ as a subset of $A = A \cap \mathrm{cl}_X(B)$

The interior of $B \subseteq A$ as a subset of $A = A \cap \underline{\mathrm{int}}_X(B)$

$f : X \to Y$ is cts $\Leftrightarrow$ for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$

If $A \subseteq X$ and $B \subseteq Y$, then $\overline{A \times B} = \overline{A} \times \overline{B}$

$x \in \overline{A} \Leftrightarrow$ every open $\mathcal{U} \in \mathcal{T}$ that contains x intersects A.

$x \in X$ is a limit point of $A \subseteq X$ if $x \in \overline{A \setminus \{x\}}$, i.e, every open set in X that contains x hits A in a point other than x.

The set of limit points of $A = A'$ = the derived set of A

$\overline{A} = A \cup A'$.

More on continuity

$f : X \to Y$ and $g : Y \to Z$ both cts $\Rightarrow g \circ f : X \to Z$ is cts

If $X = \bigcup \mathcal{U}_a lpha$, $\mathcal{U}_a lpha \in \mathcal{T}$ for all α, and $f : X \to Y$ has $f|_{\mathcal{U}_a lpha} : \mathcal{U}_\alpha \to Y$ is cts for all α, then f is cts.

If $X = C \cup D$, C, D both closed, and $f : X \to Y$ has $f|_C : C \to Y$ and $f|_D : D \to Y$ cts, then f is cts.

In reverse: if $X = C \cup D$, C, D both closed, $f : C \to Y$ and $g : D \to Y$ are both cts, and $f(x) = g(x)$ for all $x \in C \cap D$, then $h : X \to Y$, defined by $h(x) = f(x)$ if $x \in C$, $h(x) = g(x)$ if $x \in D$, is cts. A similar statement is true for X = union of open sets.

A cts bijection $f : X \to Y$ is a *homeomorphism* if the inverse function $f^{-1} : Y \to X$ is also cts. X and Y are called *homeomorphic.* A homeo gives not only a bijection between points of the spaces, but also between the open sets in the two topologies. Homeomorphic spaces have the same topological properties.

Quotient spaces

Given an equivalence relation $\sim$ on a topological space $(X, \mathcal{T})$, its quotient $X/\sim$ is the set of equivalence classes under $\sim$. The quotient map $p : X \to X/\sim$ can be used to induce a topology on $X/\sim$; $\mathcal{U} \subseteq X/\sim$ is open $\Leftrightarrow$ $p^{-1}(\mathcal{U}) \in \mathcal{T}$. This is the *quotient topology* on $X/\sim$.

Given a quotient map $p : X \to X/\sim$ and a cts function $f : X \to Z$ with $g(a) = g(b)$ whenever $p(a) = p(b)$, then f induces a continuous map $\overline{f} : X/\sim \to Z$ with $f = \overline{f} \circ p$.

If $A \subseteq X$, we can define an equiv reln generated by $x \sim y$ if $x, y \in A$; the quotient is X/A.

If $A \subseteq X$, $B \subseteq Y$ and $h : A \to B$ is a homeo, then we have an equiv reln generated by $x \sim y$ if $h(x) = y$; quotient is $X \cup_{A=B} Y$

If $f : X \to Y$ is continuous, then we have the equiv reln on $(X \times I) \cup Y$ generated by $(x, 1) \sim f(x)$; the quotient is the mapping cylinder M_f.

Connectedness

Motivation: understand the topological property underlying the Intermediate Value Theorem:

If $f : [a, b] \to \mathbb{R}$ is cts and c is between $f(a)$ and $f(b)$, then $f(d) = c$ for some $d \in [a, b]$.

Idea: focus on when IVT <u>fails</u>: If $f : X \to \mathbb{R}$ fails IVT, then

$f^{-1}((-\infty, c)) = \mathcal{U} \in \mathcal{T}, f^{-1}((c, \infty)) = \mathcal{V} \in \mathcal{T}$ satisfy $\mathcal{U} \cup \mathcal{V} = X$, $\mathcal{U} \cap \mathcal{V} = \emptyset$, $a \in \mathcal{U}, b \in \mathcal{V}$.

Conversely, a pair of such sets allows us to build a cts $f : X \to \{0, 1\} \subseteq \mathbb{R}$ failing IVT.

A *separation* (or *disconnection*) of $(X, \mathcal{T})$ is a pair $\mathcal{U}\mathcal{V} \in \mathcal{T}$ with $\mathcal{U} \cup \mathcal{V} = X$, $\mathcal{U} \cap \mathcal{V} = \emptyset$, and $\mathcal{U}, \mathcal{V} \neq \emptyset$. X is *connected* if it admits no separation.

A subset $A \subseteq X$ is connected if $(A, \mathcal{T}_A)$ is a connected space.

$A \subseteq X$ is connected $\Leftrightarrow$ whenever $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ with $A \subseteq \mathcal{U} \cup \mathcal{V}$ and $A \cap \mathcal{U} \cap \mathcal{V} = \emptyset$, either $A \subseteq \mathcal{U}$ or $A \subseteq \mathcal{V}$.

If $A \subseteq X$ is connected and $\mathcal{U}, \mathcal{V}$ separate X, then either $A \subseteq \mathcal{U}$ or $A \subseteq \mathcal{V}$.

If $(X, \mathcal{T})$ is connected and $\mathcal{T}' \subseteq \mathcal{T}$, then $(X, \mathcal{T}')$ is connected.

If $A \subseteq X$ is connected and $f : X \to Y$ is cts, then $f(A) \subseteq Y$ is connected.

If $A_\alpha \subseteq X$ are connected $\forall\ \alpha$ and $\bigcap A_\alpha \neq \emptyset$, then $\bigcup A_\alpha$ is connected.

If $A \subseteq X$ is connected and $A \subseteq\ B \subseteq \overline{A}$, then B is connected.

If X_α are all connected, then $\prod X_\alpha$ is connected, when given the product topology.

This is false in general, when using the box topology.

The connected subsets of $\mathbb{R}$ are precisely the intervals:

$(a, b), [a, b), (a, b], [a, b], (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), \emptyset, \mathbb{R}$.

Path-connectedness

A *path* in X is a cts function $\gamma : [0, 1] \to X$. X is *path-connected* if foa $x, y \in X$ $\exists$ a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x, \gamma(1) = y$.

(X, cat) path-connected $\Rightarrow$ (X, cat) connected.

The converse is not true; there are connected spaces which are not path-connected.

If $A \subseteq X$ is path-connected and $f : X \to Y$ is cts, then $f(A) \subseteq Y$ is path-connected.

The relation $x \sim y$ if $\exists$ connected $A \subseteq X$ with $x, y \in A$ is an equivalence relation; the equivalence classes are the *connected components* $[x]$ of X.

$[x] = \bigcup\{A \subseteq X \text{ connected} : x \in A\}$ = largest connected subset containing x.

Connected components are closed subsets of X.

The relation $x \sim y$ if $\exists$ path in X joining x and y is an equivalence relation; the equivalence classes are the *path components* $[x]_p$ of X.

$[x]_p = \bigcup\{A \subseteq X \text{ path connected} : x \in A\}$ = largest path connected subset containing x.

$[x]_p \subseteq [x]$; each $[x]$ is a disjoint union of $[y]_p$'s.