Math 970 Homework and Midterm problems

1. Show that if \( f : X \to Y \) is a function, then the inverse image of subsets of \( Y \) satisfies:
   \( a \) \( f^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} f^{-1}(U_i) \)
   \( b \) \( f^{-1}(\bigcap_{j \in J} V_j) = \bigcap_{j \in J} f^{-1}(V_j) \)
   \( c \) \( f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \)

2. With notation as in problem \# 1, show, by contrast, that some of the corresponding results for the image of subsets of \( X \) do not hold in general. Under what conditions of the function \( f \) would each property that fails actually hold true?

3. Show that if \( f : (X, d) \to (Y, d') \) is a function between metric spaces which satisfies, for some \( K \in \mathbb{R} \), \( d'(f(x), f(y)) \leq K \cdot d(x, y) \) for all \( x, y \in X \), then \( f \) is continuous. In particular, if \( f \) decreases distances, then \( f \) is continuous.

4. Show that the metrics \( d_1 \) and \( d_2 \) on \( \mathbb{R}^n \) satisfy
   \[ d_2(\vec{x}, \vec{y}) \leq d_1(\vec{x}, \vec{y}) \leq n \cdot \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} \leq n \cdot d_2(\vec{x}, \vec{y}) \]

   Conclude that \( d_1 \) and \( d_2 \) give the same open sets for \( \mathbb{R}^n \).

5. Show that if \( (X, d) \) is a metric space, then \( (X, \bar{d}) \), where
   \[ \bar{d}(x, y) = \min\{d(x, y), 1\} \]

   is also a metric space, with the same open sets as \( (X, d) \).

6. If \( (X, T) \) is a topological space, \( Y \) is a set, and \( f : X \to Y \) is a function, show that
   \[ T' = \{U' \subseteq Y : f^{-1}(U') \in T\} \]

   is the finest topology on \( Y \) for which \( f : (X, T) \to (Y, T') \) is continuous.

   (Note that this problem is actually asking you to show three things...)

7. If \( (X, T) \) is a topological space, and \( A \subseteq X \), then \( A \in T \) if and only if
   for all \( x \in A \), there is a \( U \in T \) so that \( x \in U \subseteq A \)

8. Show that \( B = \{(a, \infty) \times (b, \infty) : a, b \in \mathbb{R}\} \) is a basis for a topology \( T \) on \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \), which is coarser than the usual Euclidean topology on \( \mathbb{R}^2 \). Show that \( B' = \{[a, \infty) \times [b, \infty) : a, b \in \mathbb{R}\} \) is a basis for a topology \( T' \) which is strictly finer than \( T \), and not comparable to the usual Euclidean topology.

9. Show that, in general, if \( B \) and \( B' \) are both bases for topologies on \( X \), that \( B \cap B' \) and \( B \cup B' \) need not be. Show, however, that \( B'' = \{B \cap B' : B \in B, B' \in B'\} \) is a basis for a topology, and \( T(B'') \) is the coarsest topology containing both \( B \) and \( B' \).

10. Show that the topology generated by a basis \( B \) is the coarsest topology containing \( B \) (i.e., it is the intersection of all such topologies).

11. Let \( (X, T) \) be a topological space, \( B \subseteq X \) a subset, and \( T_B \) the subspace topology on \( B \). If \( A \subseteq B \), show that the subspace topology that it inherits from \( B \) is the same as the subspace topology that it inherits from \( X \).

12. Show that if \( A \subseteq X \) and \( (X, T) \) is Hausdorff, then the subspace topology on \( A \) is Hausdorff.
13. Show that if \((X,d)\) and \((Y,d')\) are metric spaces, then the product topology on \(X \times Y\) is metrizable. [There are lots of (correct) choices of metric on \(X \times Y\); you can take your cue from \(\mathbb{R}^2\).]

14. Show that if \((X,T),(Y,T')\) are topological spaces and \(x_0 \in X\), then the function 
\[ \iota_{x_0} : Y \to X \times Y , \iota_{x_0}(y) = (x_0,y) \]
is continuous.

15. Show that if \((X,d)\) is a metric space, then the metric \(d : X \times X \to \mathbb{R}\) is continuous (where \(X \times X\) has the product topology). Show, further, that the metric topology \(T\) is the coarsest topology on \(X\) for which \(d\) is continuous.

(Hint: show that if \(T' \subseteq T\), then \(N_d(x_0,\epsilon) \notin T'\) for some \(x_0\) and \(\epsilon\); now look at problem \# 14.)

16. Show that, if \(X\) is an infinite set, then the finite complement topology \(T_f\) on \(X \times X\) is not a product topology, i.e., there do not exist topologies \(T, T'\) on \(X\) whose product topology is \(T_f\). On the other hand, if \(X\) is finite, show that \(T_f\) on \(X \times X\) is a product topology.

(Hint: the basis for the product topology would have to be \(\subseteq T_f\)...)

17. For \(A, B \subseteq X\) with \((X,T)\) a topological space, if \(A\) is open in \(X\) and \(B\) is closed in \(X\), then \(A \setminus B\) is open and \(B \setminus A\) is closed.

18. Show that if \(A, B \subseteq X\), then
\[ (a) \quad \overline{A \cup B} = \overline{A} \cup \overline{B} \]
\[ (b) \quad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \text{ but that equality does not hold in general,} \]
\[ (c) \quad \overline{A \setminus B} \supseteq \overline{A} \setminus \overline{B}, \text{ but that equality does not hold in general.} \]

19. Show that if \(A \subseteq X\) and \(X\) has two topologies \(T \subseteq T'\), then if \(x \in X\) is a limit point of \(A\) w.r.t. \(T'\), then it is a limit point of \(A\) w.r.t. \(T\).

20. Show that if \(A_i \subseteq X_i\) for all \(i \in I\), then 
\[ \prod_i A_i = \prod_i \overline{A_i} \subseteq \prod_i X_i \]
for both the product and box topologies.

21. Find the closure of the set \((0,1) \subseteq \mathbb{R}\), when \(\mathbb{R}\) has the
\[ (a) \text{ finite complement topology} \]
\[ (b) \text{ infinite (open) ray to the right topology} \]
\[ (c) \text{ discrete topology} \]
\[ (d) \text{ lower limit topology}, \text{ generated by the basis } B = \{(a,b) : a, b \in \mathbb{R}\} \]

22. Show that if \(X\) is a space with topology generated by a basis \(B\), then \(X\) is Hausdorff if and only if for every \(x, y \in X\) with \(x \neq y\), there are \(B, B'\) in \(B\) with \(x \in B\), \(y \in B'\) and \(B \cap B' = \emptyset\).

23. Show that if \(T\) is the usual topology on \(\mathbb{R}\), the space \(X = \mathbb{R} \cap \{\ast\}\), with topology generated by the basis \(B = T \cup \{(U \setminus 0) \cup \{\ast\} : U \in T \text{ and } 0 \in U\}\) is not Hausdorff, but every one-point subset of \(X\) is closed. [FYI: \(X\) is called the line with two origins.]
24. Show that the line with two origins is the quotient of two disjoint copies of \( \mathbb{R} \) (think: \( \mathbb{R} \times \{0, 1\} \)). Conclude that the quotient of a Hausdorff space need not be Hausdorff.

25. Show that the quotient space obtained by the equivalence relation \( \sim \) on \([0, 1] \times [0, 1]\) generated by (i.e., add \( a \sim a \), and \( a \sim b \) whenever \( b \sim a \), and any relation that transitivity would force on you)

\[
(0, y) \sim (1, y) \text{ for all } y \in [0, 1] \text{ and } (x, 0) \sim (x, 1) \text{ for all } x \in [0, 1]
\]

admits a continuous bijection to \( S^1 \times S^1 \).

26. Find an example of subspaces \( A, B \subseteq \mathbb{R} \) (giving \( \mathbb{R} \) the usual topology) for which there is a continuous bijection

\[
f : A \to B
\]

whose inverse is not continuous.

27. Show that if \( T \subseteq T' \) are topologies on \( X \) and \( (X, T') \) is connected, then so is \( (X, T) \).

28. Find an example of a space \( X \) and subset \( A \subseteq X \) where \( \text{int}(A) \) and \( \text{cl}(A) \) are both connected, but \( A \) is not.

29. Show by example that for \( f : X \to Y \) continuous and \( A \subseteq Y \), having one of \( f^{-1}(A) \) and \( A \) connected does not necessarily imply that the other is connected.

30. Show that if \( X_\alpha, \alpha \in I \) are all path-connected, then so is \( \prod_{\alpha \in I} X_\alpha \), if we use the product topology.

31. Show that if \( A_\alpha \subseteq X, \alpha \in I \) are all path-connected, and \( \bigcap_{\alpha \in I} A_\alpha \neq \emptyset \), then \( \bigcup_{\alpha \in I} A_\alpha \) is path-connected.

32. Show that if \( C \subseteq \mathbb{R}^3 \) is countable, then \( \mathbb{R}^3 \setminus C \) is path-connected. (Hint: a plane in \( \mathbb{R}^3 \) will hit \( C \) in how many points?)

33. Show that if \( (X, T) \) is compact, and \( T' \subseteq T \), then \( (X, T') \) is compact.

34. Show that if \( (X, T) \) is a topological space and \( A, B \) are compact subsets of \( X \), then \( A \cup B \) is compact.

35. Give an example of a space \( (X, T) \) and subsets \( A, B \subseteq X \) so that \( A \) and \( B \) are compact but \( A \cap B \) is not.

(Note: your space \( X \) cannot be Hausdorff...)

36. Let \( X = \mathbb{R} \) with the infinite ray topology

\[
T = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}
\]

Show that \( A = \{0\} \) is a compact subset of \( X \), but its closure \( \bar{A} \) isn't.

37. Show that if \( (X, T) \) is a Hausdorff space and \( A, B \subseteq X \) are disjoint compact subsets of \( X \), then there are subsets \( U, V \in T \) so that \( A \subseteq U, B \subseteq V \), and \( U \cap V = \emptyset \).

38. Show that if \( X \) is limit point compact, and \( A \) is a closed subset of \( X \), then \( A \) is limit point compact.

39. Give an example of a limit point compact space \( X \) and a continuous function \( f : X \to Y \) for which \( f(X) \subseteq Y \) is not limit point compact.
Note: for the purposes of the following problems, “regular” means points and (disjoint) closed sets can be separated with open sets, “normal” means disjoint closed sets can be separated, $T_3$ means $T_1$ and regular, and $T_4$ means $T_1$ and normal.

40. Show by examples that the continuous image of $T_3$ need not be $T_3$, and that the continuous image of a non-$T_3$ space can be $T_3$!

41. Show that every closed subset of a normal space is normal, and that every closed subset of a $T_4$ space is $T_4$.

42. Show that for any collection $X_\alpha \neq \emptyset$ of topological spaces, if $\prod_\alpha X_\alpha$ is $T_4$ in the product topology, then $X_\alpha$ is $T_4$ for all $\alpha$.
   (Hint: embed $X_\alpha$ in $\prod_\alpha X_\alpha$ as a closed subset!)

43. Show that a compact metric space $(X,d)$ is second countable.
   (Hint: look at $\{N_d(x,1/n) : x \in X\}$ for each $n$.)

44. Show that a closed subset of a Lindelöf space is Lindelöf.

45. Show by example that a closed subset of a separable space need not be separable.

46. Show that the continuous image of a separable space is separable, and the continuous image of a Lindelöf space is Lindelöf.

47. Show that if $(X,T(C))$ is second countable (with $C = \{C_n\}_{n=1}^\infty$ countable), then every basis $B$ for $T = T(C)$ contains a countable basis $B' \subseteq B$.
   (Hint: look at all $B \in B$ with $C_m \subseteq B \subseteq C_n$ for some $m,n$; then pick (at most) one for each pair...)

48. Show that if $X$ is Hausdorff and $f:X \to X$ is continuous, then the fixed point set
   \[ \text{Fix}(f) = \{x \in X : f(x) = x\} \]
   of $f$ is a closed subset of $X$.

A subset $A \subseteq X$ is a retract of $X$ if there is a continuous map $r:X \to A$ with $r \circ i = Id$, i.e., $r(a) = a$ for all $a \in A$. The map $r$ is called a retraction.

49. Show that if $X$ is Hausdorff and $A \subseteq X$ is a retract of $X$, then $A$ is closed.
   (Hint: show that $A$ is the fixed point set of some map!)

50. Show that if $r : X \to A$ is a retraction and $a \in A$, then
   \[ r_* : \pi_1(X,a) \to \pi_1(A,a) \]
   is a surjective homomorphism.

51. Show that if $a \in A \subseteq X$, $\pi_1(X,a) = \{1\}$, and $f : (A,a) \to (Y,b)$ is continuous, then if $f$ extends to a continuous map $g : X \to Y$ (i.e., $g|_A = f$), then $f_* : \pi_1(A,a) \to \pi_1(Y,b)$ is the trivial homomorphism.
   (The contrapositive of the last part of this statement sounds stronger....)

52. A space $X$ is contractible if the identity map $I:X \to X$ is homotopic to the constant map $c(x) = x_0$. Show that if $X$ is contractible then any two maps $f,g : Y \to X$ are homotopic. Show that this implies that $\pi_1(X,x_0) = \{1\}$. 
M1. For $X$ any set, and $a, b \in X$, show that the collection

$$T = \{ A \subseteq X : a \in A \text{ or } b \notin A \}$$

forms a topology on $X$.

M2. Show that if $(X, T)$ and $(Y, T')$ are both Hausdorff, then $X \times Y$, with the product topology, is Hausdorff.

M3. Show that if $T \subseteq T'$ are topologies on $X$, and $A \subseteq X$, then $\text{cl}_{T'}(A) \subseteq \text{cl}_T(A)$ . Show, further, that if $T \neq T'$ then there is an $A \subseteq X$ with $\text{cl}_{T'}(A) \neq \text{cl}_T(A)$ .

M4. For $X = \mathbb{R}$, let $T_1$ be the excluded point topology, excluding 0, and let $T_2$ be the included point topology, including 1. Show that if a continuous function

$$f : (X, T_1) \rightarrow (X, T_2)$$

has $f(0) = 1$, then $f$ is constant. Show, more generally, that any continuous function

$$f : (X, T_1) \rightarrow (X, T_2)$$

has image consisting of at most 2 points.

M5. Show that if $T \subseteq T'$ are topologies on the set $X$ and $(X, T')$ is path-connected, then $(X, T)$ is path-connected.