Math 970 Homework and Midterm problems

1. Show that if $f{:}X \to Y$ is a function, then the inverse image of subsets of Y satisfies:

(a) $f^{-1}(\bigcup_{i\in I} \mathcal{U}_i) = \bigcup_{i\in I} f^{-1}(\mathcal{U}_i)$

(b) $f^{-1}(\bigcap_{j\in J} \mathcal{V}_j) = \bigcap_{j\in J} f^{-1}(\mathcal{V}_j)$

(c) $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$

2. With notation as in problem # 1, show, by contrast, that some of the corresponding results for the *image* of subsets of X do *not* hold in general. Under what conditions of the function f would each property that fails actually hold true?

3. Show that if $f{:}(X,d) \to (Y,d')$ is a function between metric spaces which satisfies, for some $K \in \mathbb{R}$, $d'(f(x),f(y)) \leq K \cdot d(x,y)$ for all $x,y \in X$, then f is continuous. In particular, if f decreases distances, then f is continuous.

4. Show that the metrics d_1 and d_2 on $\mathbb{R}^n$ satisfy
$$d_2(\vec{x},\vec{y}) \leq d_1(\vec{x},\vec{y}) \leq n{\cdot}\max\{|x_1-y_1|,\dots|x_n-y_n|\} \leq n \cdot d_2(\vec{x},\vec{y})$$
Conclude that d_1 and d_2 give the same open sets for $\mathbb{R}^n$.

5. Show that if (X,d) is a metric space, then $(X,\bar{d})$, where
$$\bar{d}(x,y) = \min\{d(x,y),1\}$$
is also a metric space, with the *same* open sets as (X,d) .

6. If $(X,\mathcal{T})$ is a topological space, Y is a set, and $f : X \to Y$ is a function, show that
$$\mathcal{T}' = \{\mathcal{U}' \subseteq Y : f^{-1}(\mathcal{U}') \in \mathcal{T}\}$$
is the finest topology on Y for which $f : (X,\mathcal{T}) \to (Y,\mathcal{T}')$ is continuous.
(Note that this problem is actually asking you to show <u>three</u> things...)

7. If $(X,\mathcal{T})$ is a topological space, and $A \subseteq X$, then $A \in \mathcal{T}$ if and only if
$$\text{for all } x \in A, \text{ there is a } U \in \mathcal{T} \text{ so that } x \in U \subseteq A$$

8. Show that $\mathcal{B} = \{(a,\infty)\times(b,\infty) : a,b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$, which is coarser than the usual Euclidean topology on $\mathbb{R}^2$. Show that $\mathcal{B}' = \{[a,\infty)\times[b,\infty) : a,b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}'$ which is strictly finer than $\mathcal{T}$, and not comparable to the usual Euclidean topology.

9. Show that, in general, if $\mathcal{B}$ and $\mathcal{B}'$ are both bases for topologies on X, that $\mathcal{B}\cap\mathcal{B}'$ and $\mathcal{B}\cup\mathcal{B}'$ <u>need</u> <u>not</u> <u>be</u>. Show, however, that $\mathcal{B}'' = \{B\cap B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$ <u>is</u> a basis for a topology, and $\mathcal{T}(\mathcal{B}'')$ is the coarest topology containing both $\mathcal{B}$ and $\mathcal{B}'$.

10. Show that the topology generated by a basis $\mathcal{B}$ is the coarsest topology <u>containing</u> $\mathcal{B}$ (i.e., it is the intersection of all such topologies).

11. Let $(X,\mathcal{T})$ be a topological space, $B \subseteq X$ a subset, and $\mathcal{T}_B$ the subspace topology on B. If $A \subseteq B$, show that the subspace topology that it inherits from B is the <u>same</u> as the subspace topology that it inherits from X.

12. Show that if $A \subseteq X$ and $(X,\mathcal{T})$ is Hausdorff, then the subspace topology on A is Hausdorff.

13. Show that if (X, d) and (Y, d') are metric spaces, then the product topology on $X \times Y$ is metrizable. [There are lots of (correct) choices of metric on $X \times Y$; you can take your cue from $\mathbb{R}^2$.]

14. Show that if $(X, \mathcal{T}), (Y, \mathcal{T}')$ are topological spaces and $x_0 \in X$, then the function

$$\iota_{x_0} : Y \to X \times Y \ , \ \iota_{x_0}(y) = (x_0, y)$$

is continuous.

15. Show that if (X, d) is a metric space, then the metric $d : X \times X \to \mathbb{R}$ is continuous (where $X \times X$ has the product topology). Show, further, that the metric topology $\mathcal{T}$ is the <u>coarsest</u> topology on X for which d is continuous.

(Hint: show that if $\mathcal{T}' \subsetneq \mathcal{T}$, then $N_d(x_0, \epsilon) \notin \mathcal{T}'$ for some x_0 and ϵ ; now look at problem # 14.)

16. Show that, <u>if</u> X is an infinite set, then the finite complement topology $\mathcal{T}_f$ on $X \times X$ is <u>not</u> a product topology, i.e., there do not exist topologies $\mathcal{T}, \mathcal{T}'$ on X whose product topology is $\mathcal{T}_f$. On the other hand, if X is finite, show that $\mathcal{T}_f$ on $X \times X$ <u>is</u> a product topology.

(Hint: the <u>basis</u> for the product topology would have to be $\subseteq \mathcal{T}_f$ ...)

17. For $A, B \subseteq X$ with $(X, \mathcal{T})$ a topological space, if A is open in X and B is closed in X, then $A \setminus B$ is open and $B \setminus A$ is closed.

18. Show that if $A, B \subseteq X$, then

   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   (b) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, but that equality does not hold in general,

   (c) $\overline{A \setminus B} \supseteq \overline{A} \setminus \overline{B}$, but that equality does not hold in general.

19. Show that if $A \subseteq X$ and X has two topologies $\mathcal{T} \subseteq \mathcal{T}'$, then if $x \in X$ is a limit point of A w.r.t. $\mathcal{T}'$, then it is a limit point of A w.r.t. $\mathcal{T}$.

20. Show that if $A_i \subseteq X_i$ for all $i \in I$, then

$$\overline{\prod_i A_i} = \prod_i \overline{A_i} \subseteq \prod_i X_i$$

for <u>both</u> the product and box topologies.

21. Find the closure of the set $(0, 1) \subseteq \mathbb{R}$, when $\mathbb{R}$ has the

   (a) finite complement topology

   (b) infinite (open) ray to the right topology

   (c) discrete topology

   (d) *lower limit topology*, generated by the basis $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$

22. Show that if X is a space with topology generated by a basis $\mathcal{B}$, then X is Hausdorff if and only if for every $x, y \in X$ with $x \neq y$, there are $B, B' \in \mathcal{B}$ with $x \in B$, $y \in B'$ and $B \cap B' = \emptyset$.

23. Show that if $\mathcal{T}$ is the usual topology on $\mathbb{R}$, the space $X = \mathbb{R} \cap \{*\}$, with topology generated by the basis $\mathbb{B} = \mathcal{T} \cup \{(U \setminus 0) \cup \{*\} : U \in \mathcal{T} \text{ and } 0 \in U\}$ is not Hausdorff, but every one-point subset of X is closed. [FYI: X is called the *line with two origins.*]

24. Show that the line with two origins is the quotient of two disjoint copies of $\mathbb{R}$ (think: $\mathbb{R} \times \{0,1\}$). Conclude that the quotient of a Hausdorff space need not be Hausdorff.
25. Show that the quotient space obtained by the equivalence relation $\sim$ on $[0,1] \times [0,1]$ generated by (i.e., add $a \sim a$, and $a \sim b$ whenever $b \sim a$, and any relation that transitivity would *force* on you)

    $$(0,y) \sim (1,y) \text{ for all } y \in [0,1] \text{ and } (x,0) \sim (x,1) \text{ for all } x \in [0,1]$$

    admits a continuous bijection to $S^1 \times S^1$.
26. Find an example of subspaces $A, B \subseteq \mathbb{R}$ (giving $\mathbb{R}$ the usual topology) for which there is a continuous bijection

    $$f : A \to B$$

    whose inverse is **not** continuous.
27. Show that if $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on X and $(X, \mathcal{T}')$ is connected, then so is $(X, \mathcal{T})$.
28. Find an example of a space X and subset $A \subseteq X$ where $\text{int}(A)$ and $\text{cl}(A)$ are both connected, but A is not.
29. Show by example that for $f : X \to Y$ continuous and $A \subseteq Y$, having one of $f^{-1}(A)$ and A connected does <u>not</u> necessarily imply that the other is connected.
30. Show that if $X_\alpha, \alpha \in I$ are all path-connected, then so is $\prod_{\alpha \in I} X_\alpha$, if we use the product topology.
31. Show that if $A_\alpha \subseteq X, \alpha \in I$ are all path-connected, and $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is path-connected.
32. Show that if $C \subseteq \mathbb{R}^3$ is countable, then $\mathbb{R}^3 \setminus C$ is path-connected. (Hint: a plane in $\mathbb{R}^3$ will hit C in how many points?)
33. Show that if $(X, \mathcal{T})$ is compact, and $\mathcal{T}' \subseteq \mathcal{T}$, then $(X, \mathcal{T}')$ is compact.
34. Show that if $(X, \mathcal{T})$ is a topological space and A, B are compact subsets of X, then $A \cup B$ is compact.
35. Give an example of a space $(X, \mathcal{T})$ and subsets $A, B \subseteq X$ so that A and B are compact but $A \cap B$ is not.

    (Note: your space X cannot be Hausdorff....)
36. Let $X = \mathbb{R}$ with the infinite ray topology

    $$\mathcal{T} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

    Show that $A = \{0\}$ is a compact subset of X, but its closure $\bar{A}$ <u>isn't</u>.
37. Show that if $(X, \mathcal{T})$ is a Hausdorff space and $A, B \subseteq X$ are disjoint compact subsets of X, then there are subsets $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ so that $A \subseteq \mathcal{U}$, $B \subseteq \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$.
38. Show that if X is limit point compact, and A is a closed subset of X, then A is limit point compact.
39. Give an example of a limit point compact space X and a continuous function $f : X \to Y$ for which $f(X) \subseteq Y$ is not limit point compact.

Note: for the purposes of the following problems, " regular" means points and (disjoint) closed sets can be separated with open sets, "normal" means disjoint closed sets can be separated, T_3 means T_1 and regular, and T_4 means T_1 and normal.

40. Show by examples that the continuous image of T_3 need not be T_3, and that the continuous image of a non-T_3 space can be T_3!

41. Show that every closed subset of a normal space is normal, and that every closed subset of a T_4 space is T_4.

42. Show that for any collection $X_\alpha \neq \emptyset$ of topological spaces, if $\prod_\alpha X_\alpha$ is T_4 in the product topology, then X_α is T_4 for all α.

    (Hint: embed X_α in $\prod_\alpha X_\alpha$ as a closed subset!)

43. Show that a compact metric space (X, d) is second countable.

    (Hint: look at $\{N_d(x, 1/n) : x \in X\}$ for each n .)

44. Show that a closed subset of a Lindelöf space is Lindelöf.

45. Show by example that a closed subset of a separable space need not be separable.

46. Show that the continuous image of a separable space is separable, and the continuous image of a Lindelöf space is Lindelöf..

47. Show that if $(X, \mathcal{T}(\mathcal{C}))$ is second countable (with $\mathcal{C} = \{C_n\}_{n=1}^\infty$ countable), then <u>every</u> basis $\mathcal{B}$ for $\mathcal{T} = \mathcal{T}(\mathcal{C})$ contains a countable basis $\mathcal{B}' \subseteq \mathcal{B}$.

    (Hint: look at all $B \in \mathcal{B}$ with $C_m \subseteq B \subseteq C_n$ for some m, n ; then pick (at most) one for each pair...)

48. Show that if X is Hausdorff and $f{:}X \to X$ is continuous, then the fixed point set

$$\text{Fix}(f) = \{x \in X : f(x) = x\}$$

    of f is a closed subset of X.

    A subset $A \subseteq X$ is a <u>retract</u> of X if there is a continuous map $r{:}X \to A$ with $r \circ i = Id$, i.e., $r(a) = a$ for all $a \in A$. The map r is called a <u>retraction</u>.

49. Show that if X is Hausdorff and $A \subseteq X$ is a retract of X, then A is closed.

    (Hint: show that A is the fixed point set of some map!)

50. Show that if $r : X \to A$ is a retraction and $a \in A$, then

$$r_*{:}\pi_1(X, a) \to \pi_1(A, a)$$

    is a surjective homomorphism.

51. Show that if $a \in A \subseteq X$, $\pi_1(X, a) = \{1\}$, and $f{:}\ \ (A, a) \to (Y, b)$ is continuous, then if f extends to a continuous map $g{:}X \to Y$ (i.e., $g|_A = f$), then $f_*{:}\pi_1(A, a) \to \pi_1(Y, b)$ is the trivial homomorphism.

    (The contrapositive of the last part of this statement sounds stronger....)

52. A space X is <u>contractible</u> if the identity map $I{:}X \to X$ is homotopic to the constant map $c(x) = x_0$. Show that if X is contractible then any two maps $f, g\ :\ Y \to X$ are homotopic. Show that this implies that $\pi_1(X, x_0) = \{1\}$.

M1. For X any set, and $a, b \in X$, show that the collection

$$\mathcal{T} = \{A \subseteq X : a \in A \text{ or } b \notin A\}$$

forms a topology on X.

M2. Show that if $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are both Hausdorff, then $X \times Y$, with the product topology, is Hausdorff.

M3. Show that if $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on X, and $A \subseteq X$, then $\text{cl}_{\mathcal{T}'}(A) \subseteq \text{cl}_{\mathcal{T}}(A)$. Show, further, that if $\mathcal{T} \neq \mathcal{T}'$ then there is an $A \subseteq X$ with $\text{cl}_{\mathcal{T}'}(A) \neq \text{cl}_{\mathcal{T}}(A)$.

M4. For $X = \mathbb{R}$, let $\mathcal{T}_1$ be the excluded point topology, excluding 0, and let $\mathcal{T}_2$ be the included point topology, including 1. Show that if a continuous function

$$f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

has $f(0) = 1$, then f is constant. Show, more generally, that any continuous function $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ has image consisting of at most 2 points.

M5. Show that if $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X and $(X, \mathcal{T}')$ is path-connected, then $(X, \mathcal{T})$ is path-connected.