Math 970 Homework

Due: Sept. 18

1. If $(X, \mathcal{T})$ is a topological space, Y is a set, and $f : X \to Y$ is a function, show that
$$\mathcal{T}' = \{\mathcal{U}' \subseteq Y : f^{-1}(\mathcal{U}') \in \mathcal{T}\}$$
is the finest topology on Y for which $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous.
(Note that this problem is actually asking you to show three things...)

2. If $(X, \mathcal{T})$ is a topological space, and $A \subseteq X$, then $A \in \mathcal{T}$ if and only if
$$\text{for all } x \in A, \text{ there is a } U \in \mathcal{T} \text{ so that } x \in U \subseteq A$$

3. Show that $\mathcal{B} = \{(a, \infty) \times (b, \infty) : a, b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, which is coarser than the usual Euclidean topology on $\mathbb{R}^2$. Show that $\mathcal{B}' = \{[a, \infty) \times [b, \infty) : a, b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}'$ which is strictly finer than $\mathcal{T}$, and not comparable to the usual Euclidean topology.

4. Show that, in general, if $\mathcal{B}$ and $\mathcal{B}'$ are both bases for topologies on X, that $\mathcal{B} \cap \mathcal{B}'$ and $\mathcal{B} \cup \mathcal{B}'$ need not be. Show, however, that $\mathcal{B}'' = \{B \cap B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$ is a basis for a topology, and $\mathcal{T}(\mathcal{B}'')$ is the coarest topology containing both $\mathcal{B}$ and $\mathcal{B}'$.

5. Show that the topology generated by a basis $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$ (i.e., it is the intersection of all such topologies).