Math 970 Homework

Due: (Fill in the date!)

1. Show that if $f{:}X \to Y$ is a function, then the inverse image of subsets of Y satisfies:

(a) $f^{-1}(\bigcup_{i\in I} \mathcal{U}_i) = \bigcup_{i\in I} f^{-1}(\mathcal{U}_i)$

(b) $f^{-1}(\bigcap_{j\in J} \mathcal{V}_j) = \bigcap_{j\in J} f^{-1}(\mathcal{V}_j)$

(c) $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$

2. With notation as in problem # 1, show, by contrast, that some of the corresponding results for the *image* of subsets of X do *not* hold in general. Under what conditions of the function f would each property that fails actually hold true?

3. Show that if $f{:}(X, d) \to (Y, d')$ is a function between metric spaces which satisfies, for some $K \in \mathbb{R}$, $d'(f(x), f(y)) \le K \cdot d(x, y)$ for all $x, y \in X$, then f is continuous. In particular, if f decreases distances, then f is continuous.

4. Show that the metrics d_1 and d_2 on $\mathbb{R}^n$ satisfy
$$d_2(\vec{x}, \vec{y}) \le d_1(\vec{x}, \vec{y}) \le n{\cdot}\max\{|x_1 - y_1|, \ldots |x_n - y_n|\} \le n \cdot d_2(\vec{x}, \vec{y})$$
Conclude that d_1 and d_2 give the same open sets for $\mathbb{R}^n$.

5. Show that if (X, d) is a metric space, then $(X, \bar{d})$, where
$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$
is also a metric space, with the *same* open sets as (X, d) .