Math 970 Final Exam

Due Wednesday, Dec. 17, 3:00 pm, on the instructor's desk. You should work the exam on your own, and should direct all questions and comments about the exam to the instructor only, to the extent reasonable.

Work any six of the following seven problems. If you provide solutions to all seven problems, your highest six grades will be used to compute your total grade. Each problem carries equal weight.

1. If $f, g{:}X \to Y$ are both continuous, and Y is Hausdorff, show that
$$C = \{x \in X : f(x) = g(x)\} \quad \text{is closed.}$$
Show that, with notation as above, if $A \subseteq X$ is dense and $f|_A = g|_A$, then $f = g$.

2. Show that for every topological space X, there is a connected topological space Y so that X is homeomorphic to a closed subset of Y.

3. Show that if $\mathcal{U} \subseteq \mathbb{R}^n$ is open (in the usual topology) and connected, then $\mathcal{U}$ is path-connected.
(Hint: show that, for each $x_0 \in \mathcal{U}$, $\{x \in \mathcal{U} : \exists \text{ path in } \mathcal{U} \text{ from } x_0 \text{ to } x\}$ is open.)

4. Show that if $(X, \mathcal{T})$ is compact and Hausdorff and $f{:}X \to X$ is continuous, then there is a closed subset $C \subseteq X$ so that $f(C) = C$.
(Hint: Start with $C_0 = X, C_1 = f(X) = f(C_0), \ldots$)

5. Show that if $(X, \mathcal{T})$ is connected and T_4, and X contains at least two points, then X is uncountable.

6. Show that if $(X, \mathcal{T})$ is T_3 and $C \subseteq X$ is closed, then
$$\bigcap\{\mathcal{U} \in \mathcal{T} : C \subseteq \mathcal{U}\} = C$$

7. Show that if $f{:}X \to Y$ is continuous, then the mapping cylinder
$$Z = ((X \times I) \coprod Y)/\sim$$
where $(x, 1) \sim f(x)$, is homotopy equivalent to Y .