Math 872 Problem Set 9

Starred (*) problems are due Thursday, April 23.

60. Find the (singular) homology groups of the Möbius band $M = [0,1] \times [0,1]/\{(0,t) \sim (1,1-t)\ :\ t \in [0,1]\}$, and compute the inclusion-induced maps $i_* : H_k(\partial M) \to H_k(M)$.

61. Use the computations from problem # 60 and the Mayer-Vietoris sequence to compute the (reduced) homology groups of the Klein bottle, which can be viewed as two Möbius bands with their boundary circles glued together $K^2 = M_1 \cup_{\partial M_1 = \partial M_2} M_2$.

(*) 62. [Hatcher, p.158, # 32] Show, using a Mayer-Vietoris sequence, that for the suspension $\Sigma X = cX \cup_X cX$ of a space X (the union of two cones on X glued together along the two copies of X) that $H_n(\Sigma X) \cong H_{n-1}(X)$ for every n.

63. Find an example of a pair of graphs Γ_1, Γ_2, so that one is a finite sheeted covering of the other $p : \Gamma_1 \to \Gamma_2$, but the induced map on homology $p_* : H_1(\Gamma_1) \to H_1(\Gamma_2)$ is <u>not</u> injective. [Hint: it actually almost never <u>is</u> injective...]

64. Show that the inclusion map ι, thought of as a map of pairs $\iota : (D^n, \partial D^n) \to (D^n, D^n \setminus \{0\})$, is a homotopy equivalence both of the spaces and of the subspaces, but is ňot a *homotopy equivalence of pairs*, that is, there is <u>not</u> a map

    $f : (D^n, D^n \setminus \{0\}) \to (D^n, \partial D^n)$ so that $f \circ \iota$ and $\iota \circ f$ are both homotopic, as maps of pairs, to the identity maps.

(*) 65. Show that $H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$ for every i; here $H_i(X) = 0$ if $i < 0$.

    [One approach: show that $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times D^n_+)$, where D^n_+ is the upper hemisphere of S^n (Problem #57 will help), and that $H_i(X \times S^n, X \times D^n_+) \cong H_{i-1}(X \times S^{n-1}, X \times D^{n-1}_+)$ by excision and the long exact sequence of the triple $(X \times D^{n-1}_+, X \times S^{n-1}, X \times D^n_-)$.]

    [Hatcher, p.158, # 36 gives a different approach.]

(*) 66. Show by induction on n that, for the n-*torus*, the Cartesian product $\mathbb{T}^n = (S^1)^n = S^1 \times \cdots \times S^1$ of n copies of the circle, we have for every k that $H_k(\mathbb{T}^n)$ is isomorphic to $\mathbb{Z}^{N(n,k)}$ for some $N(n,k) \in \mathbb{Z}$, and find a formula for $N(n,k)$.

    [Working out the homology of $\mathbb{T}^n$ for small values of n and looking for a pattern will probably help. Note that this says <u>find</u>, not <u>prove</u>, although a familiar formula for some familiar numbers probably constitutes a fine proof....]

67. Show that, for any collection of finitely generated abelian groups $G_1, \ldots, G_n$, there is a path-connected space X with $\widetilde{H}_i(X) \cong G_i$ for all $i = 1, \ldots, n$ and $\widetilde{H}_i(X) = 0$ for all other i.