Math 872 Problem Set 5

Starred (*) problems are due Thursday, March 5.

32. Is the subgroup H of $F(a,b)$ generated by the elements $\{aba, bba, bab, abb\} \subseteq F(a,b)$ isomorphic to a free group on 4 letters? That is, if you build a covering space $p : \widetilde{X} \to S^1 \vee S^1$ with $p_*(\pi_1(\widetilde{X})) = H$ (and you should!), is $\pi_1(\widetilde{X})$ free on four generators?

33. [Hatcher, p.79, # 8] Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering spaces of path-connected, locally path connected spaces X and Y with $\widetilde{X}$ and $\widetilde{Y}$ simply-connected. Show that if X and Y are homotopy equivalent, then $\widetilde{X}$ and $\widetilde{Y}$ are homotopy equivalent.

   [Note: the problem statement in Hatcher points to an earlier problem that might help...]

(*) 34. [Hatcher, p.79, problem # 4, sort of] Build a connected, simply connected covering space $p : \widetilde{X} \to X$ of the wedge of a circle and a 2-sphere $X = S^1 \vee S^2$.

   [Note: problem # 39 might give you some guidance on what parts of it should look like...]

(*) 35. Show that the fundamental group of the closed orientable surface Σ_2 of genus 2 is not abelian.

   [Hint: to show that for some pair of loops γ, η that $\gamma * \eta * \overline{\gamma} * \overline{\eta}$ isn't homotopically trivial, show that it (at least once!) does not lift to a loop in <u>some</u> covering space of Σ_2.]

36. A group G is called *residually finite* if for every $g \in G$ with $g \neq 1$, there is a finite group H and a homomorphism $\varphi : G \to H$ with $\varphi(g) \neq 1$. Show that G is residually finite <u>if</u> <u>and</u> <u>only</u> <u>if</u> for some (equivalently, any!) CW-complex X with $\pi_1(X, x_0) \cong G$ and any loop $\gamma : I \to X$ at x_0 with $1 \neq [\gamma] \in \pi_1(X)$, there is a finite-sheeting covering space $p : \widetilde{X} \to X$ and basepoint $\widetilde{x_0}$ over x_0 so that γ does <u>not</u> lift to a loop at $\widetilde{x_0}$.

37. Show that if a group G acts freely ($x = gx \Rightarrow g = 1$) and properly discontinuously (for all $x \in X$ there is a nbhd $\mathcal{U}$ of x such that $\{g \ : \ g(\mathcal{U}) \cap \mathcal{U} \neq \emptyset\}$ is finite) on a space X, then the quotient map $p : X \to X/G = X/\{x \sim gx$ for all $g \in G\}$ given by $p(x) = [x]$ is a covering map. In particular if X is Hausdorff and G is a finite group acting freely on X, then $p : X \to X/G$ is a covering map.

   [Pointless remark: some people would write our quotient space as $G \backslash X$, since G is acting on the left, and so is being quotiented out from the left, although the Wikipedia entry on the matter, *http://en.wikipedia.org/wiki/Group_action*, agrees with us in this. Besides, as one usually learns when TeXing things, TeX doesn't like \ as a symbol, it asks what the macro "\X[next symbol]" is supposed to mean ...]

(*) 38. [Hatcher, p.79, problem # 9] Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $f : X \to S^1$ is homotopic to a constant map.

39. [Hatcher, p.80, problem # 15] Suppose $p : \widetilde{X} \to X$ is a simply connected covering space of X, and $A \subseteq X$ is a path-connected, locally path-connected subspace of X, with $\widetilde{A}] \subseteq \widetilde{X}$ a path component of $p^{-1}(A)$. Show that $p_*(\pi_1(\widetilde{(A)})) \subseteq \pi_1(A)$ is the kernel of the inclusion-induced homomorphism $\pi_1(A) \to \pi_1(X)$.