Math 872 Problem Set 1

Starred (*) problems are due Thursday, January 29.

1. Show that the cone on a circle $cS^1 = S^1 \times I/\sim$, where $(x,1) \sim (y,1)$ for $x, y \in S^1$, with the quotient topology, is homeomorphic to the unit disk $D^2 = \{\vec{x} \in \mathbb{R}^2 \ : \ ||\vec{x}|| \leq 1\}$.

2. If $x_0 \in A \subseteq X$ and $\pi_1(X, x_0) = \{1\}$, show that, for any space Y, if a cts map $f : A \to Y$ extends to a map $g : X \to Y$, then $f_* : \pi_1(A, x_0) \to \pi_1(Y, f(x_0))$ is the trivial map.

(*) 3. [Hatcher, p.38, #5] Show that $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$ if and only if every cts map $f : S^1 \to X$ extends to a map $F : D^2 \to X$ (i.e., for some F, we have $F \circ \iota = f$).

4. [Hatcher, p.38, #2] Show that, in general, if $\alpha, \beta : I \to X$ are paths which are *homotopic rel endpoints*, $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1) = x_1$, then their associated change of basepoint maps are equal: $\alpha_* = \beta_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

   [N.B.: Hatcher calls these maps β_α and (sorry...) β_β, for reasons I don't understand. Maybe $\beta =$ 'basepoint'?]

5. [Hatcher, p.38, #3] Show that for a path-connected space X, $\pi_1(X)$ is abelian $\Leftrightarrow$ the change of basepoint maps are all independent of path, i.e.,

   for $\alpha, \beta : I \to X$ with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$, we always have $\alpha_* = \beta_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

   [Useful notation: our hypothesis can be expressed symbolically by saying that α, β are maps of triples; $\alpha, \beta : (I, 0, 1) \to (X, x_0, x_1)$.]

(*) 6. Show that every homomorphism $\varphi : \mathbb{Z} \to \pi_1(X, x_0)$ can be realized as the induced homomorphism $\varphi = f_*$ of a continuous map $f : (S^1, (1, 0)) \to (X, x_0)$. [Hint: Look at $\varphi(1) \in \pi_1(X, x_0)$.]

(*) 7. [Hatcher, p.39, # 13] If $x_0 \in A \subseteq X$ and A is path-connected, show that the inclusion-induced map $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective $\Leftrightarrow$ for every path $\gamma : I \to X$ with endpoints in A, $\gamma(0), \gamma(1) \in A$, γ is homotopic rel endpoints to a path in A, that is, $\gamma \simeq \alpha$ rel ∂I with $\alpha : I \to A \subseteq X$.

8. A *topological group* is a space G with continuous maps $G \times G \to G$ and $G \to G$, denoted $(g, h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$, which (together with an $e \in G$) make G a group. Show that for loops $\alpha, \beta : (I, \partial I) \to (G, e)$, the loop $\gamma(t) = \alpha(t) \cdot \beta(t)$ is homotopic, rel endpoints, to <u>both</u> $\alpha * \beta$ and $\beta * \alpha$. Conclude that for any topological group G, $\pi_1(G, e)$ is <u>abelian</u>.

   [Hint: $\alpha * \beta(t) = (\alpha * c_e)(t) \cdot (c_e * \beta)(t)$.]